A brief introduction to distributional elements

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2026.02.27

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de Rham complex

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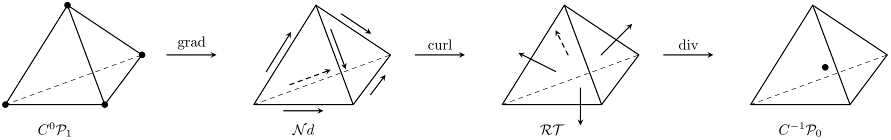
$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Whitney form



The DOFs of these finite element spaces are

- $C^0\mathcal{P}_1$: $v(x)$ for each vertex $x \in \mathcal{V}$;
- $\mathcal{N}d$: $\int_e \boldsymbol{v} \cdot t_e$ for each edge $e \in \mathcal{E}$;
- $\mathcal{RT}$: $\int_f \boldsymbol{v} \cdot n_f$ for each face $f \in \mathcal{F}$;
- $C^{-1}\mathcal{P}_0$: $\int_K v$ for each element $K \in \mathcal{T}_h$.

Distribution

A **distribution** $T : C_c^\infty(\Omega) \to \mathbb{R}$ is continuous linear functional. We write its action as:

$$\langle T, v\rangle.$$

The derivative T' of a distribution T is defined by

$$\langle T', v\rangle = -\langle T, v'\rangle.$$

Every locally integrable function $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a distribution via:

$$\langle f, v\rangle = \int_\Omega f(x)v(x)\mathrm{d}x.$$

For each simplex σ, we define the distribution δ_σ as

$$\langle \delta_\sigma . v\rangle = \int_\sigma v.$$

Particularly, for each vertex $x \in \mathcal{V}$, $\langle \delta_x, v\rangle = v(x)$.

Distributional de Rham complex

Consider distributional elements

$$0 \to C^0\mathcal{P}_1 \xrightarrow{\text{grad}} \mathcal{N}d \xrightarrow{\text{curl}} \mathcal{RT}^{-1} \xrightarrow{\text{div}} C^{-2}\mathcal{P}_0 \to 0$$

Here

$$\mathcal{RT}^{-1} := \{v \in L^2 : v|_K \in \mathcal{RT}(K)\}$$

and any $q \in C^{-2}\mathcal{P}_0$ is distribution such that for any $v \in C_c^\infty(\Omega)$

$$\langle q, v\rangle = \sum_{K\in\mathcal{T}_h} \int_K q_K v + \sum_{f\in\mathring{\mathcal{F}}} \int_f q_f v$$

with $q_K, q_f \in \mathbb{R}$. Then any $q \in C^{-2}\mathcal{P}_0$ can be represented as

$$q = \sum_{K\in\mathcal{T}_h} q_K\delta_K + \sum_{f\in\mathring{\mathcal{F}}} q_f\delta_f.$$

Here div : $\mathcal{RT}^{-1} \to C^{-2}\mathcal{P}_0$ is distributional derivative such that for any $v \in C_c^\infty(\Omega)$

$$\begin{aligned} \langle \operatorname{div} \sigma, v \rangle &= -\langle \sigma, \operatorname{grad} v \rangle = -\int_\Omega \sigma \cdot \operatorname{grad} v \\ &= \sum_{K \in \mathcal{T}_h} \Big(\int_K \operatorname{div} \sigma_K v - \sum_{f \subset \partial K} \int_f (\sigma_K \cdot n_f) v \Big) \end{aligned}$$

Then for any $\sigma \in \mathcal{RT}^{-1}$, $q = \operatorname{div} \sigma \in C^{-2}\mathcal{P}_0$ satisfies

$$q_K = \operatorname{div} \sigma_K, \quad q_f = -\sum_{f \subset \partial K} (\sigma_K \cdot n_f) = [\sigma \cdot n_f]_f$$

Here $[v]_f$ denotes the jump of v across the face of f.

Exactness of distributional complex

$$0 \to C^0\mathcal{P}_1 \xrightarrow{\text{grad}} \mathcal{N}d \xrightarrow{\text{curl}} \mathcal{RT}^{-1} \xrightarrow{\text{div}} C^{-2}\mathcal{P}_0 \to 0$$

Exactness:

- for any $q = \sum_{K\in\mathcal{T}_h} q_K\delta_K + \sum_{f\in\mathring{\mathcal{F}}} q_f\delta_f \in C^{-2}\mathcal{P}_0$, we first define $\sigma^1 \in \mathcal{RT}^{-1}$ such that for any

$$\sigma_K^1 \cdot n_f = -\tfrac{1}{2}q_f, \quad \forall K \in \mathcal{T}_h, f \subset \partial f$$

  Next choose $\sigma^2 \in \mathcal{RT}$ such that $\operatorname{div}\sigma_K^2 = q_K - \operatorname{div}\sigma_K^1$. Then set $\sigma = \sigma^1 + \sigma^2 \in \mathcal{RT}^{-1}$ with

$$\operatorname{div}\sigma = q.$$

- for any $\sigma \in \mathcal{RT}^{-1}$, $\operatorname{div}\sigma = 0$ implies that

$$\operatorname{div}\sigma_K = 0, \quad \forall K \in \mathcal{T}_h, \qquad [\sigma\cdot n_f]_f = 0, \quad \forall f \in \mathring{\mathcal{F}}$$

  Then $\sigma \in \mathcal{RT}$ with $\operatorname{div}\sigma = 0$, hence $\sigma = \operatorname{curl} u$ with some $u \in \mathcal{N}d$.

Other distributional complexes:

$$0 \to C^0\mathcal{P}_1 \xrightarrow{\text{grad}} \mathcal{N}d^{-1} \xrightarrow{\text{curl}} \mathcal{RT}^{-2} \xrightarrow{\text{div}} C^{-3}\mathcal{P}_0 \to 0$$

$$0 \to C^{-1}\mathcal{P}_1 \xrightarrow{\text{grad}} \mathcal{N}d^{-2} \xrightarrow{\text{curl}} \mathcal{RT}^{-3} \xrightarrow{\text{div}} C^{-4}\mathcal{P}_0 \to 0$$

Here for any $q \in C^{-4}\mathcal{P}_0$

$$\langle q, v\rangle = \sum_{K\in\mathcal{T}_h}\int_K q_K v + \sum_{f\in\mathring{\mathcal{F}}}\int_f q_f v + \sum_{e\in\mathring{\mathcal{E}}}\int_e q_e v + \sum_{x\in\mathring{\mathcal{V}}} q_e v(x)$$

for any $\sigma \in \mathcal{N}d^{-2}$

$$\langle \sigma, v\rangle = \sum_{K\in\mathcal{T}_h}\int_K \sigma_K \cdot v + \sum_{f\in\mathring{\mathcal{F}}}\int_f \sigma_f \cdot v$$

here $\sigma_K \in \mathcal{N}d(K)$ and $\sigma_f \in P_1(f) \otimes n_f$

High order distributional complex

$$0 \to C^{-1}\mathcal{P}_{k+1} \xrightarrow{\text{grad}} \mathcal{N}d_k^{-2} \xrightarrow{\text{curl}} \mathcal{RT}_k^{-3} \xrightarrow{\text{div}} C^{-4}\mathcal{P}_k \to 0$$

For any $u \in C^{-1}\mathcal{P}_{k+1}$ and $v \in [C_c^\infty(\Omega)]^3$

$$\begin{aligned}\langle \operatorname{grad} u, v\rangle &= -\langle u, \operatorname{div} v\rangle = -\int_\Omega u \operatorname{div} v \\ &= \sum_{K\in\mathcal{T}_h}\int_K \operatorname{grad} u \cdot v - \sum_{f\in\mathring{\mathcal{F}}}\int_f u n \cdot v\end{aligned}$$

then $\operatorname{grad} u = \sum_{K\in\mathcal{T}_h} \operatorname{grad} u \delta_K - \sum_{f\in\mathring{\mathcal{F}}} u n_f \delta_f \in \mathcal{N}d_k^{-2}$

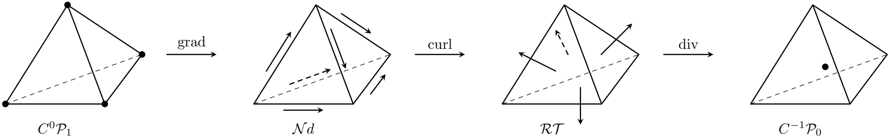
For any $\sigma = \sum_{K\in\mathcal{T}_h} \sigma_K \delta_K + \sum_{f\in\mathring{\mathcal{F}}} \sigma_f n_f \delta_f$ with $\sigma_K \in \mathcal{N}d_k(K)$ and $\sigma_f \in P_{k+1}(f)$

$$
\begin{aligned}
\langle \operatorname{curl}\sigma, v\rangle &= \langle \sigma, \operatorname{curl} v\rangle \\
&= \sum_{K\in\mathcal{T}_h} \int_K \sigma_K \cdot \operatorname{curl} v - \sum_{f\in\mathring{\mathcal{F}}} \int_f \sigma_f n_f \cdot \operatorname{curl} v \\
&= \sum_{K\in\mathcal{T}_h} \int_K \sigma_K \cdot \operatorname{curl} v - \sum_{f\in\mathring{\mathcal{F}}} \int_f \sigma_f \operatorname{rot}_f v \\
&= \sum_{K\in\mathcal{T}_h} \int_K \operatorname{curl}\sigma_K \cdot v + \sum_{f\in\mathring{\mathcal{F}}} \int_f (\operatorname{curl}_f \sigma_f - (\sigma_K \times n_f)) \cdot v \\
&\qquad - \sum_{e\in\mathring{\mathcal{E}}} \int_e \sigma_f t_e \cdot v
\end{aligned}
$$

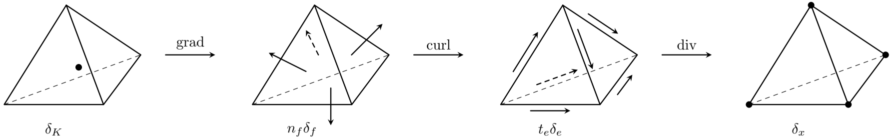
Then

$$
\operatorname{curl}\sigma = \sum_{K\in\mathcal{T}_h} \operatorname{curl}\sigma_K \delta_K + \sum_{f\in\mathring{\mathcal{F}}} (\operatorname{curl}_f \sigma_f - (\sigma_K \times n_f))\delta_f - \sum_{e\in\mathring{\mathcal{E}}} \sigma_f t_e \delta_e \in \mathcal{RT}_k^{-3}
$$

Whitney form



Distributional de Rham complex is dual complex of Whitney form:



Application

Poisson problem:

$$-\Delta u = f, \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega.$$

And let $u_h \in V_h := C^0\mathcal{P}_1$ be the finite element solution for linear elements

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in V_h$$

A posteriori error estimates: For any $\sigma \in H(\operatorname{div})$ with $\operatorname{div}\sigma + f = 0$, there holds that

$$\|\nabla u - \nabla u_h\|_{L^2}^2 + \|\nabla u - \sigma\|_{L^2}^2 = \|\nabla u_h - \sigma\|_{L^2}^2$$

Note that $\nabla u_h \notin H(\operatorname{div})$. Assume that f is piecewise constant, then possible construction of σ is from mixed method: Find $\sigma \in \mathcal{RT}$ with $\hat{u}_h \in C^{-1}\mathcal{P}_0$ such that

$$\begin{cases} (\sigma, \tau_h) + (\hat{u}_h, \operatorname{div}\tau_h) = 0, & \forall \tau_h \in \mathcal{RT}, \\ (\operatorname{div}\sigma, v_h) = -(f, v_h), & \forall v_h \in C^{-1}\mathcal{P}_0. \end{cases}$$

Equilibration: Consider $\sigma^\Delta := \sigma - \nabla u_h \in \mathcal{RT}^{-1}$. The condition $\sigma^\Delta + \nabla u_h \in H(\mathrm{div})$ and $\mathrm{div}(\sigma^\Delta + \nabla u_h) = -f$ are rewritten as

$$\begin{aligned} \mathrm{div}\,\sigma_K^\Delta &= -f, \quad \text{in each element } K \\ [\sigma^\Delta \cdot n_f]_f &= -[\nabla u_h \cdot n_f]_f, \quad \text{on each face } f. \end{aligned}$$

This is equivalent to

$$\mathrm{div}\,\sigma^\Delta = \mathrm{div}\,\sigma - \mathrm{div}\,\nabla u_h = -(f + \Delta u_h), \quad \text{in distribution.}$$

If f is not piecewise constant, then let $\bar{f}$ be the L^2-projection of f onto piecewise constant functions. There holds the a posteriori error estimate

$$c_0 \|\sigma^\Delta\|_{L^2} - ch\|f - \bar{f}\|_{L^2} \le \|\nabla u - \nabla u_h\|_{L^2} \le \|\sigma^\Delta\|_{L^2} + ch\|f - \bar{f}\|_{L^2}.$$

Similar results hold for the curl-curl equation: Find $u \in H(\mathrm{curl})$ such that

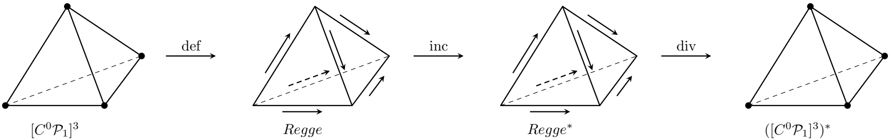
$$(\mu^{-1}\,\mathrm{curl}\,u, \mathrm{curl}\,v) = (j, v), \quad \forall v \in H(\mathrm{curl}).$$

Elasticity complex

$$0 \to [H^1(\Omega)]^3 \xrightarrow{\text{def}} H(\text{inc};\mathbb{S}) \xrightarrow{\text{inc}} H(\text{div};\mathbb{S}) \xrightarrow{\text{div}} [L^2(\Omega)]^3 \to 0$$

Here $\text{def} = \text{sym}\,\nabla$ (deformation operator), $\text{inc} = \text{curl}^T\,\text{curl}$ (incompatibility operator).

Regge-elasticity complex



$$\text{Regge} := \{\sigma \in L^2(\Omega;\mathbb{S}) : \sigma|_K \in P_0(K;\mathbb{S}), t_e^T\sigma t_e \text{ is continuous on each edge } e\}$$

$$\text{Regge}^* = \text{span}\{t_e t_e^T \delta_e\}$$

For any $\sigma \in$ Regge, it can be shown that $\operatorname{curl}^T \operatorname{curl} \sigma \in \text{Regge}^*$: For any matrix function τ, $\tau \times n, \operatorname{curl} \tau$ acts row-wise, $n \times \tau, \operatorname{curl}^T \tau$ acts columnwise, then for any symmetric smooth field $v \in C_c^\infty(\Omega; \mathbb{S})$

$$
\begin{aligned}
\langle \operatorname{curl}^T \operatorname{curl} \sigma, v \rangle &= \langle \sigma, \operatorname{curl} \operatorname{curl}^T v \rangle = \sum_{K \in \mathcal{T}_h} \int_K \sigma \cdot \operatorname{curl} \operatorname{curl}^T v \\
&= -\sum_{K \in \mathcal{T}_h} \sum_{f \subset \partial K} \int_f (\sigma \times n_f) \cdot \operatorname{curl}^T v \\
&= \sum_{K \in \mathcal{T}_h} \sum_{f \subset \partial K} \int_f (n_f \times \sigma) \cdot \operatorname{curl} v \\
&= \sum_{f \in \mathring{\mathcal{F}}} \int_f [(n_f \times \sigma)]_f \cdot \operatorname{curl} v = \sum_{f \in \mathring{\mathcal{F}}} \int_f [(n_f \times \sigma) n_f]_f \cdot (\operatorname{curl} v \cdot n_f) \\
&= \sum_{f \in \mathring{\mathcal{F}}} \sum_{e \subset \partial f} \int_e ([(n_f \times \sigma) n_f]_f t_e^T) \cdot v
\end{aligned}
$$

here we use that $[n_f \times \sigma \times n_f]_f = 0$.

Then

$$\operatorname{curl}^T \operatorname{curl} \sigma = \sum_{e \in \mathring{\mathcal{E}}} a_e m_e t_e^T$$

with some vector m_e. Since $\operatorname{curl}^T \operatorname{curl} \sigma$ is symmetric, we can get that

$$\operatorname{curl}^T \operatorname{curl} \sigma = \sum_{e \in \mathring{\mathcal{E}}} a_e t_e t_e^T \delta_e$$

with

$$a_e = \sum_{e \subset \partial f} ([n_{f,e}^T \sigma n_f]_f)$$

The generalized Regge elements $Regge^r$

$$Regge^r := \{\sigma \in L^2(\Omega; \mathbb{S}^n) : \sigma_K \in P_r(K; \mathbb{S}^n), K \in \mathcal{T}_n, \text{ tangetial-tangetial part } \sigma_{tt} \text{ is continuous}\}$$

The DOFS are: for each k-face f with $k \geq 1$,

$$\Sigma_f := \{\int_f \Pi_f \sigma \Pi_f \cdot q | q \in P_{r-k+1}(f; \mathbb{S}^k)\}$$

Application

- Finite element approximations on manifolds (Curvature, Einstein tensor, etc.)
- Continuum mechanics

Tangential-Displacement and Normal-Normal-Stress

Linear elasticity:

$$A\sigma = \varepsilon(u), \quad \operatorname{div}\sigma = -f \quad \text{ in } \Omega,$$

Here the vector field u is the unknown displacement with $u|_{\partial\Omega} = 0$, $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the strain, and σ represents the symmetric stress tensor.

Primal mixed variational formulation: Find $u \in [H_0^1(\Omega)]^d$ and $\sigma \in L^2(\Omega; \mathbb{S}^d)$ such that

$$\begin{aligned} \int_\Omega A\sigma \cdot \tau - \int_\Omega \varepsilon(u) \cdot \tau &= 0, \quad \forall \tau \in L^2(\Omega; \mathbb{S}^d) \\ -\int_\Omega \varepsilon(v) \cdot \sigma &= -\int_\Omega f \cdot v, \quad \forall v \in [H_0^1(\Omega)]^d \end{aligned}$$

Dual mixed variational formulation: Find $u \in [L^2(\Omega)]^d$ and $\sigma \in H(\operatorname{div}, \Omega; \mathbb{S}^d)$ such that

$$\begin{aligned} \int_\Omega A\sigma \cdot \tau + \int_\Omega u \cdot \operatorname{div}\tau &= 0, \quad \forall \tau \in H(\operatorname{div}, \Omega; \mathbb{S}^d) \\ \int_\Omega \operatorname{div}\sigma \cdot v &= -\int_\Omega f \cdot v, \quad \forall v \in [L^2(\Omega)]^d \end{aligned}$$

The second line corresponds to the equilibrium equation

$$b(\sigma, v) = \langle \operatorname{div} \sigma, v \rangle_{V^* \times V} = \langle -f, v \rangle_{V^* \times V}, \quad \forall v \in V$$

For primal mixed formulation $V = [H_0^1(\Omega)]^d$, and for dual mixed formulation $V = [L^2(\Omega)]^d$.

TDNNS: $V = H_0(\text{curl})$, the dual space is

$$H^{-1}(\text{div}) = \{q \in H^{-1}, \operatorname{div} q \in H^{-1}\}$$

For a tensor field σ, $\operatorname{div} \sigma \in H^{-1}(\text{div})$ is equivalent to $\sigma \in \Sigma = H(\operatorname{div}\operatorname{div})$:

$$H(\operatorname{div}\operatorname{div}) := \{\sigma \in L^2(\Omega; \mathbb{S}^d) | \operatorname{div}\operatorname{div} \sigma \in H^{-1}\}$$

The TDNNS mixed formulationis: Find $u \in H_0(\text{curl})$ and $\sigma \in \Sigma$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) = 0, &\quad \forall \tau \in \Sigma, \\ b(\sigma, v) = -\langle f, v \rangle, &\quad \forall v \in H_0(\text{curl}) \end{aligned}$$

with $a(\sigma, \tau) = \int_\Omega A\sigma \cdot \tau$ and

$$\begin{aligned} b(\sigma, \tau) &= \sum_{K \in \mathcal{T}_h} \Big(\int_K \operatorname{div} \tau \cdot v - \sum_{f \subset \partial K} \tau_{nt} \cdot v_t \Big) \\ &= \sum_{K \in \mathcal{T}_h} \Big(- \int_K \tau \cdot \varepsilon(v) + \sum_{f \subset \partial K} \tau_{nn} \cdot v_n \Big) \end{aligned}$$

Other distributional elements

TDNNS method:

$$\Sigma_h := \{\sigma \in L^2(\Omega;\mathbb{S}) : \sigma|_K \in P_k(\Omega;\mathbb{S}),\ \text{normal-normal part } \sigma_{nn} \text{ is continuous}\}$$

is a nonconforming element of

$$H(\operatorname{div}\operatorname{div}) := \{u \in L^2, \operatorname{div}\operatorname{div} u \in H^{-1}\}$$

MCS method:

$$W_h := \{w \in L^2(\Omega;\mathbb{T}) : w|_K \in P_k(K;\mathbb{T}),\ \text{tangential-normal part } w_{tn} \text{ is continuous}\}$$

is a nonconforming element of

$$H(\operatorname{curl}\operatorname{div}) := \{u \in L^2, \operatorname{curl}\operatorname{div} u \in H^{-1}\}$$

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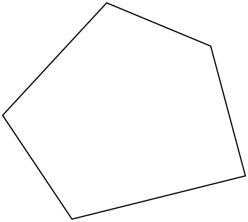
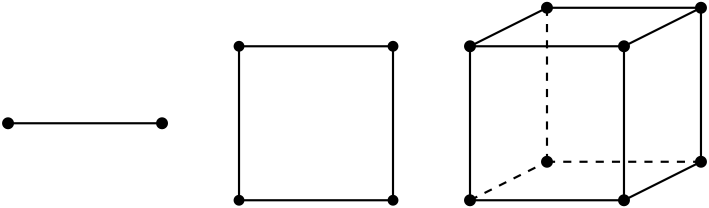
Polyhedral Mesh



- Mesh generation and adaptation: local mesh adaptation requires special strategies to either prevent or treat **hanging nodes**.
- Other reasons: non-matching grids, anisotropic mesh;
- Existing methods: Conforming finite elements(based on generalized barycentric coordinates), Virtual element methods(VEM), Discrete de Rham complex (DDR), Spline complex, ...

Rectangular mesh

Tensor product:



$P_1(x)$ $Q_{1,1}(x,y)$ $Q_{1,1,1}(x,y,z)$

de Rham complex

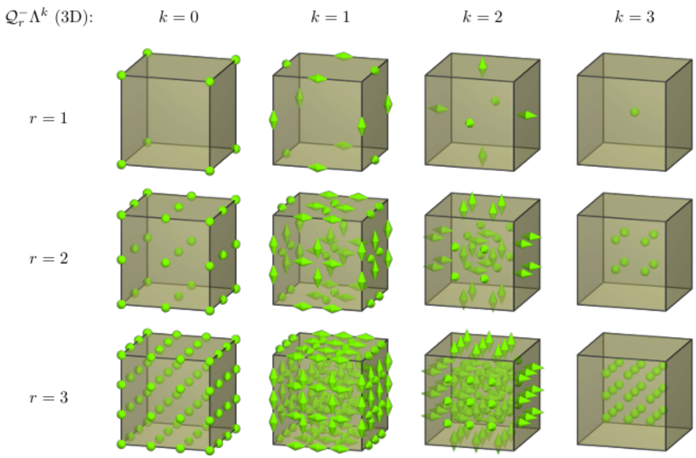


Figure: Tensor product de Rham complex, ARNOLD, DOUGLAS N., DANIELE BOFFI, and FRANCESCA BONIZZONI. "Tensor product finite element differential forms and their approximation properties." arXiv preprint arXiv:1212.6559 (2012).

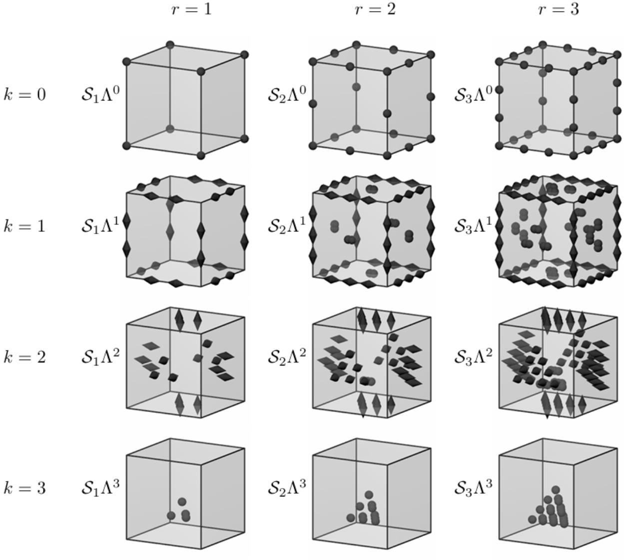
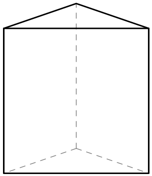


Figure: (Serendipity) de Rham complex, Arnold, Douglas, and Gerard Awanou. "Finite element differential forms on cubical meshes." Mathematics of computation 83.288 (2014): 1551-1570.

TDNNS on prism



Set $\Omega = \Omega_{\boldsymbol{x}} \times \Omega_z$, let $\mathcal{T}^{\boldsymbol{x}}$ be a shape-regular triangulation of $\Omega_{\boldsymbol{x}}$, let $\mathcal{T}^z$ be a subdivision of Ω_z, the tensor product mesh

$$\mathcal{T} = \mathcal{T}^{\boldsymbol{x}} \times \mathcal{T}^z = \{T = T^{\boldsymbol{x}} \times T^z, T^{\boldsymbol{x}} \in \mathcal{T}^{\boldsymbol{x}}, T^z \in \mathcal{T}^z\}$$

Define following finite element spaces on $\Omega_{\boldsymbol{x}}$

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{x}}^k &= \{w \in P_{\boldsymbol{x}}^k(\mathcal{T}^{\boldsymbol{x}}) : w \text{ continuous } \}\\
\mathcal{N}_{\boldsymbol{x}}^k &= \{c \in [P_{\boldsymbol{x}}^k(\mathcal{T}^{\boldsymbol{x}})]^2 : w_t \text{ continuous } \}\\
\Sigma_{\boldsymbol{x}}^k &= \{\sigma \in [P_{\boldsymbol{x}}^k(\mathcal{T}^{\boldsymbol{x}})]^{2\times 2}_{\text{sym}} : \sigma_{nn} \text{ continuous } \}\\
\mathcal{P}_{\boldsymbol{x}}^k &= \ P_{\boldsymbol{x}}^k(\mathcal{T}^{\boldsymbol{x}})
\end{aligned}
$$

Continuous and non-continuous finite element spaces on Ω_z:

$$
\begin{aligned}
\mathcal{L}_z^k = \Sigma_z^k &= \{w \in P_z^k(\mathcal{T}^{\boldsymbol{z}}) : w \text{ continuous } \}\\
\mathcal{P}_z^k = \mathcal{N}_z^k &= \ P_z^k(\mathcal{T}^z)
\end{aligned}
$$

Tensor product finite element spaces

$$
V_k = \{v \in [L^2(\Omega)]^3 : v_{\boldsymbol{x}} \in \mathcal{N}_{\boldsymbol{x}}^k \times \mathcal{L}_z^{k+1}, v_z \in \mathcal{L}_{\boldsymbol{x}}^{k+1} \times \mathcal{N}_z^k, v_t = 0 \text{ on } \Gamma_D\}
$$

$$
\begin{aligned}
\Sigma_k = \{\sigma \in L^2_{\text{sym}}(\Omega) : \sigma_{\boldsymbol{x}} \in \Sigma_{\boldsymbol{x}}^k \times \mathcal{P}_z^{k+1}, \sigma_{\boldsymbol{x}z} \in \mathcal{P}_{\boldsymbol{x}}^k \times \mathcal{P}_z^k,\\
\sigma_z \in \mathcal{P}_{\boldsymbol{x}}^{k+1} \times \Sigma_z^{k+1}, \sigma_{nn} = 0 \text{ on } \Gamma_n\}
\end{aligned}
$$

For any $v \in V_k$ and $\sigma \in \Sigma_k$

$$v = \begin{bmatrix} v_{\boldsymbol{x}} \\ v_z \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{\boldsymbol{x}} & \sigma_{\boldsymbol{x}z} \\ \sigma_{\boldsymbol{x}z} & \sigma_z \end{bmatrix}$$

Then v_t and σ_{nn} is continuous. The discrete problem is to find: $v \in V_K$, $\sigma \in \Sigma_k$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= 0, \quad \forall \tau \in \Sigma_k, \\ b(\sigma, v) &= -\langle f, v \rangle, \quad \forall v \in V_k. \end{aligned}$$

Tensor product Regge elements: for any

$$R = \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix}$$

we want R_{tt} to be continuous, then

$$R_x \in \mathcal{P}_x^{k_1} \times \mathcal{L}_y^{k_2}, \quad R_y \in \mathcal{L}_x^{k_3} \times \mathcal{P}_y^{k_4}, \quad R_{xy} \in \mathcal{P}_x^{k_5} \times \mathcal{P}_y^{k_6},$$