Grad–Shafranov: MHD equilibria and how to find them

Thomas Higham
Mathematical Institute
University of Oxford

December 2025

Contents

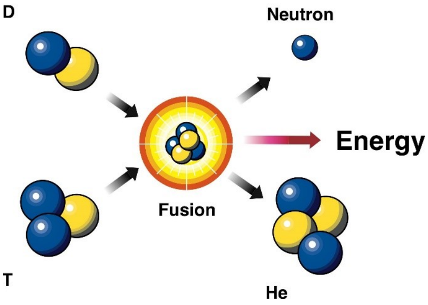
- MHD (magnetohydrodynamic) equilibria - justifying tokamaks.
- Grad-Shafranov equation.
- Helicity-preserving FE scheme of Mingdong et al He et al. 2025.
- 3D MRX Paper Blickhan, Stratton, and Kaptanoglu 2025.

Blickhan, Tobias, Julianne Stratton, and Alan A. Kaptanoglu (2025). *MRX: A differentiable 3D MHD equilibrium solver without nested flux surfaces*. arXiv: 2510.26986 [physics.comp-ph]. URL: https://arxiv.org/abs/2510.26986.

He, Mingdong et al. (2025). *Helicity-preserving finite element discretization for magnetic relaxation*. arXiv: 2501.11654 [math.NA]. URL: https://arxiv.org/abs/2501.11654.

Fusion

- Fusion of nuclei can be exothermic reaction - use to heat water.
- High pressure plasma tries to expand out - we have to confine with magnetic fields.
- To sustain reaction we want plasma in equilibrium.

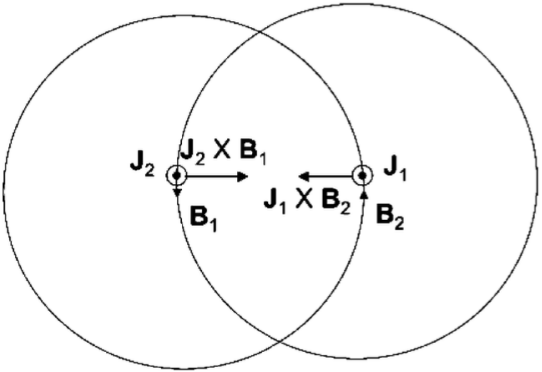


Set-up

- The following exposition follows chapters 15 and 18 in Schnack 2012.
- We assume ideal MHD setting.
- Time dependent PDEs for MHD can be derived from first principles - I avoid these for now and focus on equilibrium expressions.

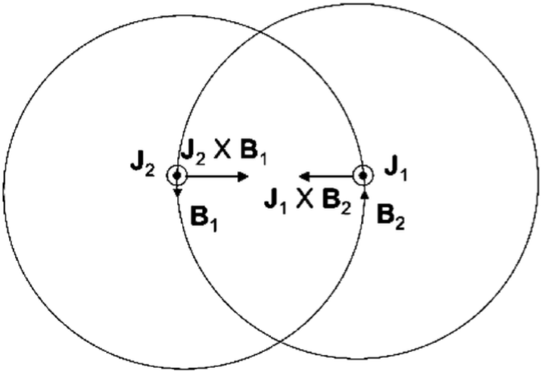
Schnack, Dalton D. (2012). *Lectures in Magnetohydrodynamics - With an Appendix on Extended MHD*. 1st. Heidelberg: Springer Berlin. ISBN: 978-3-642-26921-9.

MHD Equilibria



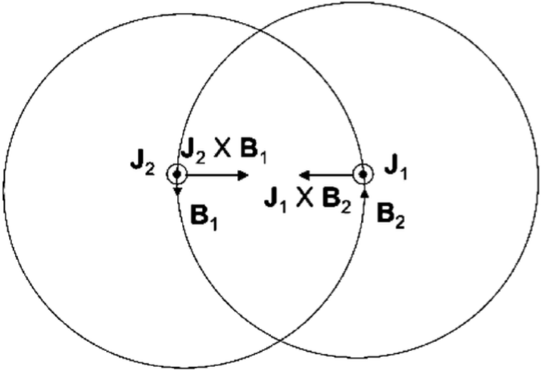
- Parallel currents are attractive.

MHD Equilibria



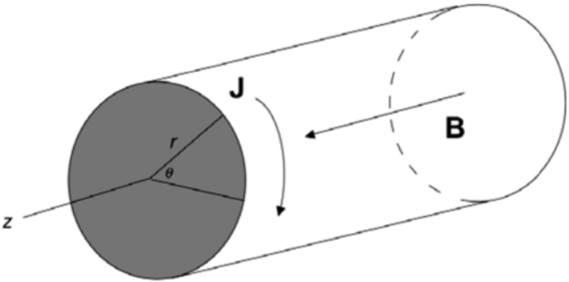
- Parallel currents are attractive.
- Net effect of Lorenz forces is compressing the fluid.
- Pinch until force balance: $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$.

MHD Equilibria



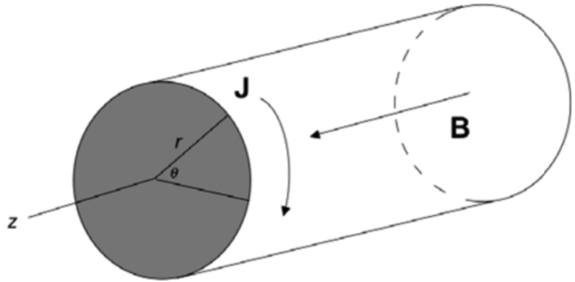
- Parallel currents are attractive.
- Net effect of Lorenz forces is compressing the fluid.
- Pinch until force balance: $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$.
- Ampère's law $\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B}$.
- Gauss' law $\nabla \cdot \boldsymbol{B} = 0$.

Toy Problem 1 - Theta-pinch



- Infinite cylinder.
- Current flows in negative θ-direction, straight magnetic field in z-direction.
- $\boldsymbol{J} \times \boldsymbol{B} = -J_\theta B_z \boldsymbol{e_r}$.

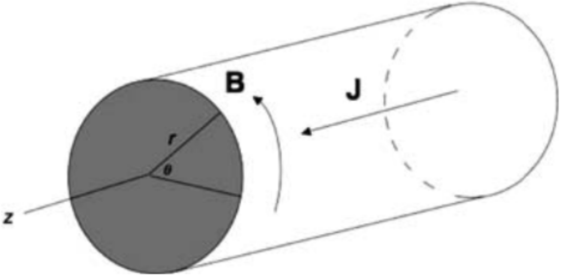
Toy Problem 1 - Theta-pinch



- Infinite cylinder.
- Current flows in negative θ-direction, straight magnetic field in z-direction.
- $\boldsymbol{J} \times \boldsymbol{B} = -J_\theta B_z \boldsymbol{e_r}$.
- Using Gauss' and Ampère's laws we can show

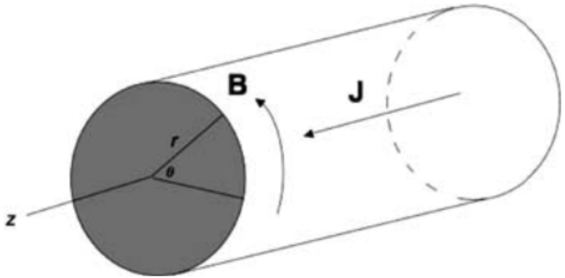
$$p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}.$$

Toy Problem 2 - z-pinch



- Current flows in z-direction, curved magnetic field in θ-direction.
- $\boldsymbol{J} \times \boldsymbol{B} = -J_z B_\theta \boldsymbol{e_r}$.

Toy Problem 2 - z-pinch



- Current flows in z-direction, curved magnetic field in θ-direction.
- $\boldsymbol{J} \times \boldsymbol{B} = -J_{z} B_{\theta} \boldsymbol{e_r}$.
- $\mu_0 \boldsymbol{J} = \nabla \times \boldsymbol{B}$ gives

$$\mu_0 \boldsymbol{J}_z = \frac{1}{r}\frac{d}{dr}(rB_\theta).$$

- We can show

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) = -\frac{B_\theta^2}{\mu_0 r}.$$

RHS is Hoop stress.

General Screw Pinch

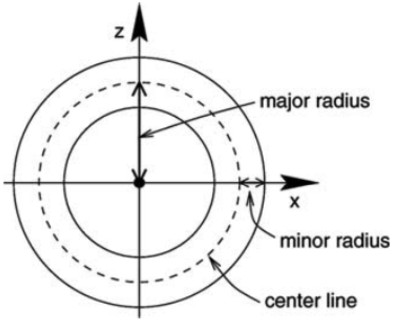
- The magnetic and current field lines wrap around the cylinder in a helical fashion.
- Now we can show equilibrium is given by

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) = \frac{B_\theta^2}{\mu_0 r}.$$

- One equation, three unknowns: we can use two things to control equilibrium.

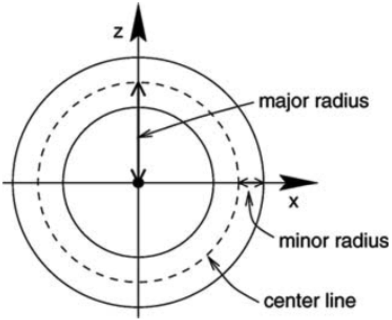
From Cylinder to Torus

UNIVERSITY OF OXFORD
Mathematical Institute



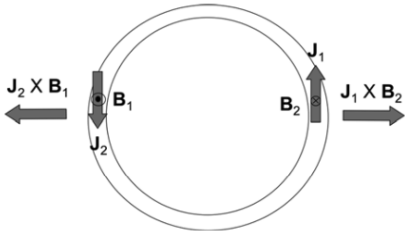
- We work in cylindrical coordinates (R, ϕ, Z).
- We have axisymmetry (no dependence on ϕ).

From Cylinder to Torus



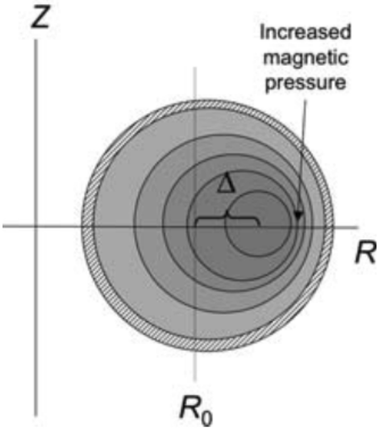
- We work in cylindrical coordinates (R, ϕ, Z).
- We have axisymmetry (no dependence on ϕ).
- We break MHD equilibrium and get outwards expansion of plasma.

Outward Expansion



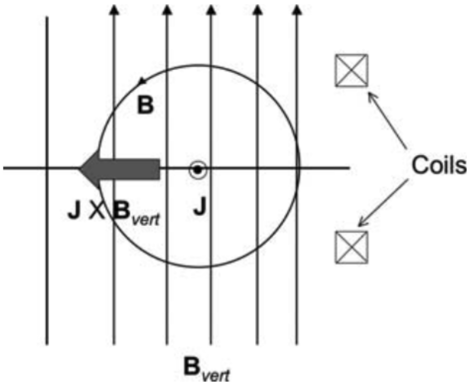
- Reason 1: outside surface area is larger in torus.
- Reason 2: antiparallel currents repel.
- External fields and currents are necessary to maintain equilibrium.

Electrical Conducting Shell



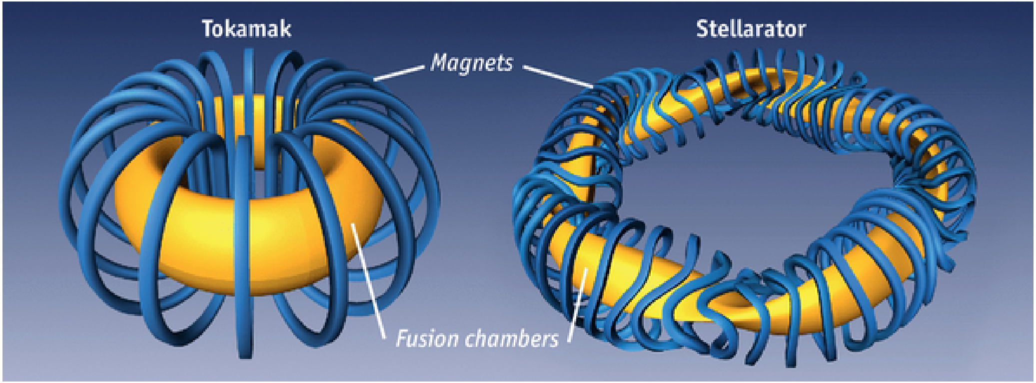
- Place perfect electrical conducting shell around minor cross section.
- As plasma expands outward, magnetic field lines enclosing the fluid won't be able to penetrate the shell so will get trapped between shell and fluid.
- Higher pressure towards "outboard side" - opposes expansion.
- New equilibrium.

Helmholtz Coils



- Place Helmholtz coils to induce a magnetic field in the z-direction.
- Interact with toroidal current (going into page).
- Inwards Lorentz force.

Stellarator



Economist.com

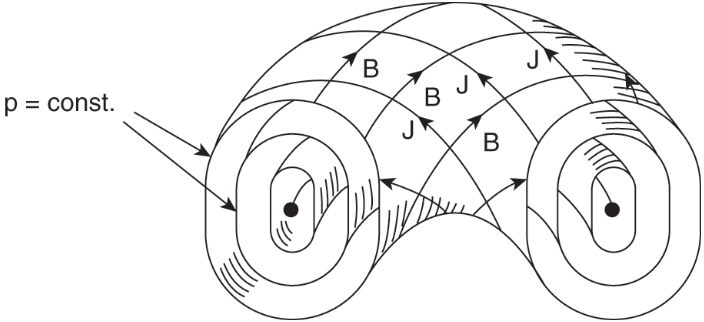
The Equilibrium Problem in an Axisymmetric Torus

- We want to derive an equation to solve, as in infinite cylinder case.
- Force balance $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$ gives us

$$\boldsymbol{B} \cdot \nabla p = 0 \quad \text{and} \quad \boldsymbol{J} \cdot \nabla p = 0,$$

  so field lines of $\boldsymbol{B}$ and current $\boldsymbol{J}$ must lie within constant pressure surfaces.
- These constant pressure surfaces form nested flux surfaces - we can label with any variable that is constant on them (we will soon use a new variable ψ).

Nested flux surfaces



Grad-Shafranov

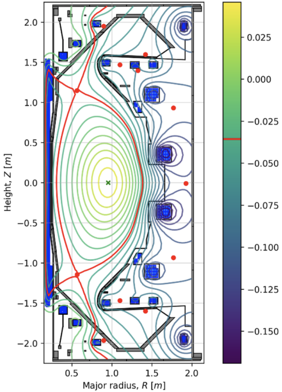
University of Oxford Mathematical Institute

- $$\Delta^*\psi = -\mu_0 R^2 p' - FF',$$
  $$\text{where } \Delta^*\psi := R\nabla \cdot \left(\frac{1}{R}\nabla\psi\right) - \frac{1}{R^2}\frac{\partial\psi}{\partial R},$$
- Go to page 112 of Schnack 2012.

Schnack, Dalton D. (2012). *Lectures in Magnetohydrodynamics - With an Appendix on Extended MHD*. 1st. Heidelberg: Springer Berlin. ISBN: 978-3-642-26921-9.

Grad-Shafranov Simulation

UNIVERSITY OF OXFORD
Mathematical Institute



- Simulation on MAST-U reactor Pentland et al. 2025.
- Free boundary problem.
- Can use Farrell's method of continuous deflation to compute a second equilibrium solution!
- Highlights why we care about fusion equilibria.

Pentland, K. et al. (2025). "Multiple solutions to the static forward free-boundary Grad-Shafranov problem on MAST-U". In: *Nucl. Fusion* 65.086053. DOI: 10.1088/1741-4326/adf3cc.

The problem with Grad-Shafranov

- We can only get nested flux surfaces. No complicated structures (magnetic islands).
- No fusion reactor is perfectly axisymmetric.
- In 2D there is no notion of helicity.

We want to do better - this is where the papers come in!