The geometric interpretation of FEEC

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Finite element exterior calculus

Differential forms is an algebraic construction, which originates in the study of smooth manifolds.[1]

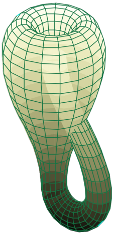

Figure: [2]Klein bottle

[1] Credited to Grassmann, Cartan, and Poincare (Loring W. Tu; An Introduction to Manifolds 2nd ed.)

[2] https://en.wikipedia.org/wiki/Manifold#/media/File:Klein_bottle.svg

Finite element exterior calculus

Differential forms can be written down explicitly, and are convenient for doing computation. Usually, they are written like

$$\sum_{i \in \mathscr{I}_k} a_i \underbrace{dx^{i_1} \wedge \cdots \wedge dx^{i_k}}_{k \text{ entries}}.$$

We can talk about 0-forms, 1-forms, all the way up to n forms.

Finite element exterior calculus

An n form encodes integration through the formula

$$\int_U a\, dx^1 \wedge \cdots \wedge dx^n := \int_U a\, dx^1 \cdots dx^n.$$

This is invariant under change of coordinates, so the definition extends to manifolds.

Finite element exterior calculus

Differential forms can be pulled back via transformations.

Figure: A path is a 1 dimensional sub-manifold.

Finite element exterior calculus

The tangential and normal traces appear as special cases of the pullback.

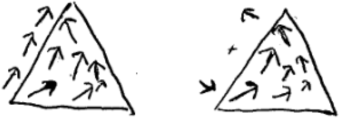

Figure: The tangential trace (left) and normal trace (right) acting on a vector field.

Finite element exterior calculus

In three dimensions the exterior derivative corresponds to classical differential operators[3]

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl},\Omega) & \xrightarrow{\text{curl}} & H(\text{div},\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} & H\Lambda^2(\Omega) & \xrightarrow{d} & H\Lambda^3(\Omega).
\end{array}$$

[3] For the necessary results on Sobolev spaces see: Gol'dshtein, Kuz'minov, and Shvedov; Differential forms on Lipschitz manifolds; 1982.

Finite element exterior calculus

It was realized that discretizing through differential forms recovers common discretizations of $H(\text{curl})$ and $H(\text{div})$. Heuristically, this is because it is the natural language for doing integration by parts.

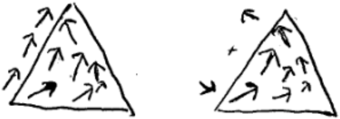

Figure: The tangential trace (left) and normal trace (right) acting on a vector field.

Finite element exterior calculus

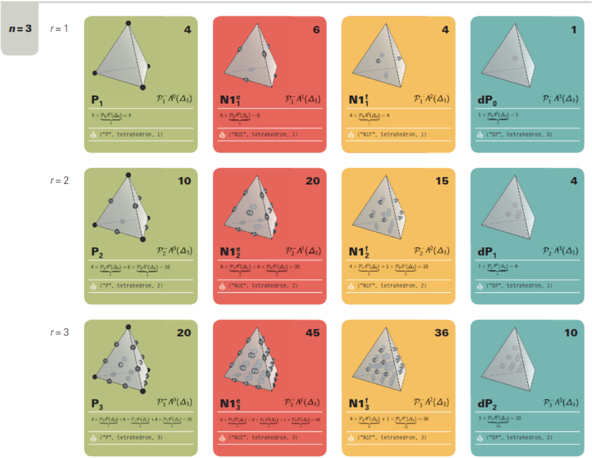

Figure: [4]Table of the trimmed polynomial forms.

[4]Picture taken from
https://www-users.cse.umn.edu/~arnold/femtable/index.html

Finite element exterior calculus

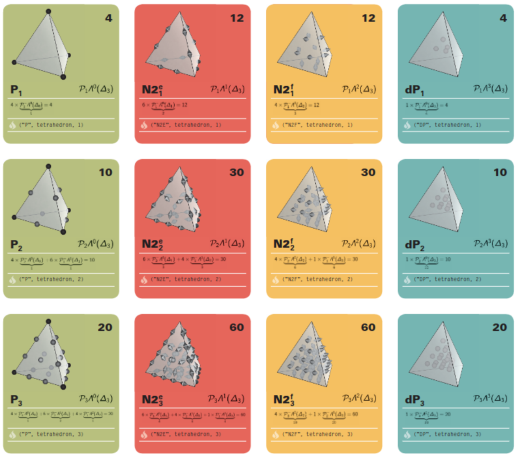

Figure: [5]Table of the full polynomial forms

[5]Picture taken from
https://www-users.cse.umn.edu/~arnold/femtable/index.html

Finite element exterior calculus

So, lets talk about discrete differential forms!

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Whitney forms

The Whitney forms are in the first row of the trimmed polynomial forms. They are the fundamental polynomial differential forms.

Figure: [6]The Whitney forms in three dimensions.

[6]Picture taken from
https://www-users.cse.umn.edu/~arnold/femtable/index.html

Whitney forms

The Whitney forms are geometric, in the sense that there exists a basis that is indexed by the set of simplices in your mesh.

$$\mathfrak{W}^k = \bigoplus_{\sigma \in \mathcal{T}^k} \lambda_\sigma .$$

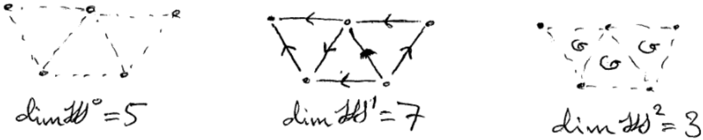

Figure: The Whitney forms are in one-to-one correspondence with the co-chains. Recall that in two dimensions there are 0, 1, and 2 -forms.

Whitney forms

Each vertex is associated with a Whitney 0-form, which is just the barycentric coordinate

$$\lambda_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

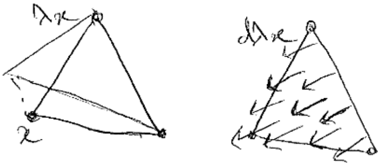

Figure: A Whitney 0-forms (left) and its gradient (right).

Whitney forms

The exterior derivative of λ_x is a constant 1-form, which is orthogonal to the facet opposite x, and is given by

$$d\lambda_x \equiv \operatorname{grad} \lambda_x.$$

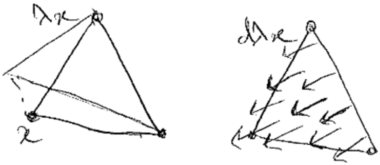

Figure: A Whitney 0-forms (left) and its gradient (right).

Whitney forms

Each edge is associated with a Whitney 1-form, which is the Nedelec curl element, or rotated Raviart-Thomas

$$\int_{\eta} \lambda_e \cdot t_{\eta} = \begin{cases} 1 & \text{if } \eta = e, \\ 0 & \text{if } \eta \neq e. \end{cases}$$

Figure: A Whitney 1-forms (left) and its curl (right).

Whitney forms

The exterior derivative of λ_e is a constant 2-form, given by

$$d\lambda_e \equiv -2\,\mathrm{Area}(T)\,dx \wedge dy.$$

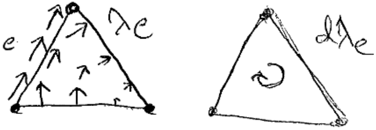

Figure: A Whitney 1-forms (left) and its curl (right).

Whitney forms

In general, each simplex σ of dimension k is associated with a Whitney k-form that satisfies, for all $\tau \in \mathcal{T}^k$

$$\int_\tau \lambda_\sigma = \begin{cases} 1 & \text{if } \tau = \sigma, \\ 0 & \text{if } \tau \neq \sigma. \end{cases}$$

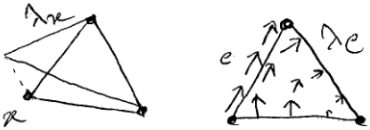

Figure: Whitney basis 0-form (left) and a Whitney basis 1-form (right).

Whitney forms

This construction admits explicit formulas!

$$\lambda_{[x_0 \cdots x_k]} = k! \sum_{i=0}^{k} (-1)^i \lambda_{x_i} d\lambda_{x_0} \wedge \cdots \widehat{d\lambda_{x_i}} \wedge \cdots d\lambda_{x_k}.$$

Whitney forms

The higher order polynomial forms can be expressed in terms of Whitney forms.

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Polynomial forms

The trimmed polynomial forms (Nédélec type 1 elements) are Whitney forms with polynomial coefficients

$$\mathcal{P}_r^- \Lambda^k(T) := \mathrm{span}\{a\lambda_\sigma : a \in \mathbb{P}_{r-1}(T),\, \sigma \in T^k\}.$$

Figure: Whitney basis 0-form (left) and a Whitney basis 1-form (right).

Polynomial forms

We adopt the notation

$$\mathcal{P}_r^- \Lambda^k(T) = \mathbb{P}_{r-1}(T) \wedge \mathfrak{W}^k(T),$$

where

$$A \wedge B := \mathrm{span}\{a \wedge b : a \in A,\ b \in B\}.$$

Polynomial forms

The full polynomial forms (Nédélec type 2 elements) consists of all the differential forms with polynomial coefficients

$$\mathcal{P}_r\Lambda^k(T) := \{a\, d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_k} : a \in \mathbb{P}_r(T),\, i \in \mathscr{I}_k\}.$$

From the fact that $d\lambda_{[i_1 \cdots i_k]} \propto d\lambda_{i_1} \wedge \cdots \wedge d\lambda_{i_k}$, we deduce the geometric interpretation

$$\mathcal{P}_r\Lambda^k(T) = \mathbb{P}_r(T) \wedge d\mathfrak{W}^{k-1}(T).$$

Polynomial forms

The polynomial forms behave particularly nicely with respect to the restriction and the exterior derivative.

Polynomial forms

For any flat simplex $\sigma \subset T$, the restriction to σ maps

- $\mathcal{P}_r^- \Lambda^k(T)$ to $\mathcal{P}_r^- \Lambda^k(\sigma)$;
- $\mathcal{P}_r \Lambda^k(T)$ to $\mathcal{P}_r \Lambda^k(\sigma)$.

In particular, the restriction of a Whitney k form to any simplex $\sigma \subset T$ of dimension k is constant!

Polynomial forms

The exterior derivative d maps

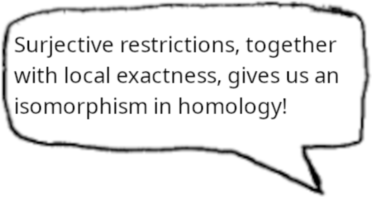
- $\mathcal{P}_r^-\Lambda^k(T)$ to $\mathcal{P}_r^-\Lambda^{k+1}(T)$;
- $\mathcal{P}_r\Lambda^k(T)$ to $\mathcal{P}_{r-1}\Lambda^{k+1}(T)$.

Moreover, the local spaces form exact sequences

$$0 \longrightarrow \mathbb{R} \hookrightarrow \mathcal{P}_r^-\Lambda^0(T) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(T) \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0(T) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(T) \longrightarrow 0.$$

Polynomial forms



Polynomial forms

The best way to work with polynomial forms on the computer is via. dof.

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Degrees of freedom

The two constructions are related via. degrees of freedom

$$\mathcal{P}_r\Lambda^k(T) \cong \bigtimes_{f \triangleleft T} \mathcal{P}^-_{r+k-\mathrm{dim}(f)}\Lambda^{\mathrm{dim}(f)-k}(f).$$

Degrees of freedom

Specifically, for $u \in \mathcal{P}_r\Lambda^k(T)$, the canonical dof are

$$u \mapsto \int_f u \wedge q \qquad \forall q \in \mathcal{P}^-_{r+k-\dim(f)}\Lambda^{\dim(f)-k}(f),\ \forall f \lhd T.$$

For $u \in \mathcal{P}^-_r\Lambda^k(T)$, they are

$$u \mapsto \int_f u \wedge q \qquad \forall q \in \mathcal{P}_{r+k-\dim(f)-1}\Lambda^{\dim(f)-k}(f),\ \forall f \lhd T.$$

Degrees of freedom

For example, $\mathcal{P}_2^- \Lambda^1(T)$ has two dof per edge, and two interior dof.

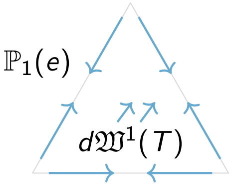


Figure: Degrees of freedom for second order Whitney forms

Degrees of freedom

The degrees of freedom encode the restriction operator, in the sense that for $i : F \hookrightarrow T$, it holds

$$\int_f \operatorname{tr}_F u \wedge q = \int_f u \wedge q \qquad \forall f \triangleleft F.$$

Degrees of freedom

The degrees of freedom encode the restriction operator, in the sense that for $i : F \hookrightarrow T$, it holds

$$\int_f \operatorname{tr}_F u \wedge q = \int_f u \wedge q \qquad \forall f \triangleleft F.$$

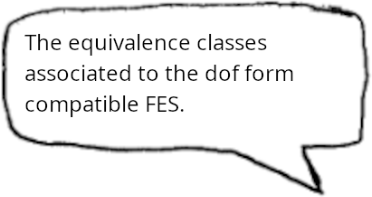
They also encode the exterior derivative through Stokes' theorem

$$\int_f du \wedge q = \int_{\partial f} u \wedge q + (-1)^{k+1} \int_f u \wedge dq.$$

Degrees of freedom

We conclude that, the degrees of freedom encode the FEEC structure, independent of the metric and basis representatives!

Degrees of freedom



Degrees of freedom

In order to make further statements about degrees of freedom we introduce the notion of bubble forms.

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Bubble forms

A discrete form is called a bubble if it has zero trace

$$\mathcal{P}_r^- \Lambda_0^k(T) := \{u \in \mathcal{P}_r^- \Lambda^k(T) : \operatorname{tr}_{\partial T} u = 0\}.$$

Bubble forms

For example, on a triangle T it holds

$$\mathcal{P}_2\Lambda^1_0(T) = \bigoplus \{\, \lambda_1\lambda_2 d\lambda_0,\ \lambda_2\lambda_0 d\lambda_1,\ \lambda_0\lambda_1 d\lambda_2 \,\}.$$

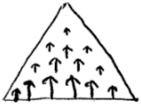

Figure: Illustration of a bubble 1-form.

Bubble forms

Bubble spaces are in one-to-one correspondence with the dof

$$\mathcal{P}_r\Lambda^k_0(T) \cong \mathcal{P}_{r+k-n}\Lambda^{n-k}(T).$$

In particular, we have the formula

$$\dim \mathcal{P}^-_r\Lambda^k(T) = \sum_{f \lhd T} \dim \mathcal{P}^-_r\Lambda^k_0(f).$$

Bubble forms

The bubble spaces also form differential complexes[7]

$$0 \longrightarrow \mathcal{P}_r^- \Lambda_0^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^- \Lambda_0^{n-1} \xrightarrow{d} \mathcal{P}_r^- \Lambda^n \longrightarrow \mathbb{R} \longrightarrow 0$$

- The exterior derivative preserves boundary conditions.
- It turns out this sequence is exact.

[7] Here we have omitted the "(T)" in order to save space. We will do this for the rest of the section.

Bubble forms

A FES that admits extensions has locally exact bubbles iff. the underlying FES is locally exact!

Bubble forms

Actually, this exactness also holds for the dof, in the sense that the bottom row of the following diagram is exact[8]

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{P}_r^-\Lambda_0^0 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{P}_r^-\Lambda_0^{n-1} & \xrightarrow{d} & \mathcal{P}_r^-\Lambda^n & \longrightarrow & \mathbb{R} & \longrightarrow & 0, \\
 & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong & & & & \\
0 & \longleftarrow & \mathcal{P}_r\Lambda^n & \xleftarrow{d} & \cdots & \xleftarrow{d} & \mathcal{P}_{r-n+1}\Lambda^1 & \xleftarrow{d} & \mathcal{P}_{r-n}\Lambda^0 & \hookleftarrow & \mathbb{R} & \longleftarrow & 0.
\end{array}$$

[8] but the diagram does not commute

Bubble forms

Clearly bubble forms are important for theory, but how difficult is it to work with them on the computer?

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Identification of bubble spaces

Recall that

$$\mathcal{P}_2\Lambda^1_0(T) = \bigoplus \{ \lambda_1\lambda_2 d\lambda_0,\ \lambda_2\lambda_0 d\lambda_1,\ \lambda_0\lambda_1 d\lambda_2 \}.$$

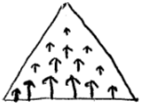


Figure: Illustration of a bubble 1-form in $\mathcal{P}_2\Lambda^1_0$.

Identification of bubble spaces

Similarly, it holds

$$\mathcal{P}_2^- \Lambda_0^1(T) = \mathrm{span}\{\, \lambda_0 \lambda_{[12]},\, \lambda_1 \lambda_{[20]},\, \lambda_2 \lambda_{[01]} \,\}.$$

Figure: Illustration of a bubble 1-form in $\mathcal{P}_2^- \Lambda_0^1$.

Identification of bubble spaces

What is the pattern on the right-hand side?

$$\mathcal{P}_2\Lambda^1_0(T) = \bigoplus \{ \lambda_1\lambda_2 d\lambda_0,\ \lambda_2\lambda_0 d\lambda_1,\ \lambda_0\lambda_1 d\lambda_2 \},$$
$$\mathcal{P}_2^-\Lambda^1_0(T) = \operatorname{span} \{ \lambda_0\lambda_{[12]},\ \lambda_1\lambda_{[20]},\ \lambda_2\lambda_{[01]} \}.$$

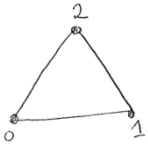


Figure: Vertices of a triangle.

Identification of bubble spaces

What is the pattern on the right-hand side?

$$\mathcal{P}_2\Lambda^1_0(T) = \bigoplus \{\, \lambda_1\lambda_2 d\lambda_0,\ \lambda_2\lambda_0 d\lambda_1,\ \lambda_0\lambda_1 d\lambda_2 \,\},$$
$$\mathcal{P}_2^-\Lambda^1_0(T) = \operatorname{span}\{\, \lambda_0\lambda_{[12]},\ \lambda_1\lambda_{[20]},\ \lambda_2\lambda_{[01]} \,\}.$$

None of the indices repeat!

Identification of bubble spaces

Given a sub-simplex $f \triangleleft T$ with vertices $x_0, \cdots, x_k$; we let $y_0, \cdots, y_{n-k-1}$ denote the vertices of T that are not in f.

$$\widehat{f} := [y_0 \cdots y_{n-k-1}].$$

Figure: On a triangle, the opposite simplex of an edge is a vertex.

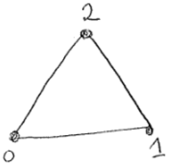
Identification of bubble spaces

The opposite scalar of f in T is defined as

$$\pi_{\widehat{f}} := \prod_{i=0}^{n-k-1} \lambda_{y_i}.$$

For example

- $\lambda_1\lambda_2 d\lambda_0 = \pi_{\widehat{[0]}} d\lambda_0$;
- $\lambda_0\lambda_{[12]} = \pi_{\widehat{[12]}}\lambda_{[12]}$.



Identification of bubble spaces

For any simplex $f \triangleleft T$, it holds that

$$\pi_{\widehat{f}} d\lambda_f, \quad \text{and} \quad \pi_{\widehat{f}} \lambda_f$$

are bubble forms.

Identification of bubble spaces

In fact, this describes all the bubbles

$$\mathcal{P}_r\Lambda_0^k(T) = \mathbb{P}_{r+k-n-1}(T) \wedge \bigoplus_{f \in T^{k-1}} \pi_{\widehat{f}}\, d\lambda_f;$$

$$\mathcal{P}_r^-\Lambda_0^k(T) = \mathbb{P}_{r+k-n-1}(T) \wedge \bigoplus_{f \in T^{k}} \pi_{\widehat{f}}\, \lambda_f.$$

Identification of bubble spaces

The proof for $\mathcal{P}_r^-\Lambda^k(T)$ is by a standard induction argument.

The proof for $\mathcal{P}_r\Lambda^k(T)$ relies on a deeper duality result.

Identification of bubble spaces

[9]For any choice of coefficients u_f, the following are equivalent

$$\sum_{f\in T^k} u_f \pi_{\widehat{f}} \lambda_f = 0,$$

$$\int_T \Big(\sum_{f\in T^k} u_f \pi_{\widehat{f}} \lambda_f\Big) \wedge \Big(\sum_{g\in T^k} \mathrm{sign}(g) u_g d\lambda_{\widehat{g}}\Big) = 0,$$

$$\sum_{g\in T^k} \mathrm{sign}(g) u_g d\lambda_{\widehat{g}} = 0.$$

[9]Christiansen, Rapetti; On high order finite element spaces of differential forms, Mathematics of computation, 2016.

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Applications

We can use the explicit characterization of the polynomial bubble spaces to design new finite element spaces, with prescribed dof.

Applications

The theory of FES tells us exactly when such a construction yields a consistent discretization.

Applications

The bubble/dof duality can be defined as a discrete Hodge star.
This could be used to

- define notions of weak continuity;
- enforce weak boundary conditions;
- do primal formulations.

Thank you!

References:

- Loring W. Tu; An Introduction to Manifolds; 2nd ed.
- Gol'dshtein, Kuz'minov, and Shvedov; Differential forms on Lipschitz manifolds; 1982.
- Christiansen, Rapetti; On high order finite element spaces of differential forms, Mathematics of computation, 2016.

Appendix

The polynomial spaces are related through the Koszul operator

$$\mathcal{P}_r^- \Lambda^k(T) = \mathcal{P}_{r-1}\Lambda^k(T) + \kappa \mathcal{P}_{r-1}\Lambda^{k+1}(T).$$

The interpretation is that $\mathcal{P}_r^- \Lambda^\bullet(T)$ is the smallest differential complex containing $\mathcal{P}_{r-1}\Lambda^\bullet(T)$ for which you can apply a Koszul argument; as pointed out by Boris.

Appendix

There is an analogous dof diagram for the full polynomial forms

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{P}_r\Lambda^0_0 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{P}_{r-n+1}\Lambda^{n-1}_0 & \xrightarrow{d} & \mathcal{P}_{r-n}\Lambda^n & \longrightarrow & \mathbb{R} & \longrightarrow & 0, \\
 & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong & & & & \\
0 & \longleftarrow & \mathcal{P}^-_{r-n}\Lambda^n & \xleftarrow{d} & \cdots & \xleftarrow{d} & \mathcal{P}^-_{r-n}\Lambda^1 & \xleftarrow{d} & \mathcal{P}^-_{r-n}\Lambda^0 & \hookleftarrow & \mathbb{R} & \longleftarrow & 0.
\end{array}$$

Appendix

One interpretation of the dof isomorphism is that of a Hodge duality

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{P}_{r+n}\Lambda^0_0 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{P}_{r+1}\Lambda^{n-1}_0 & \xrightarrow{d} & \mathcal{P}_r\Lambda^n & \longrightarrow & \mathbb{R} & \longrightarrow & 0, \\
 & & \uparrow{\scriptstyle \star_h} & & & & \uparrow{\scriptstyle \star_h} & & \uparrow{\scriptstyle \star_h} & & & & \\
0 & \longleftarrow & \mathcal{P}^-_r\Lambda^n & \xleftarrow{d} & \cdots & \xleftarrow{d} & \mathcal{P}^-_r\Lambda^1 & \xleftarrow{d} & \mathcal{P}^-_r\Lambda^0 & \hookleftarrow & \mathbb{R} & \longleftarrow & 0.
\end{array}$$

The Hodge map $\star_h : \mathcal{P}^-_r\Lambda^k(T) \to \mathcal{P}_{r+k}\Lambda^{n-k}_0(T)$ is defined by

$$\int_T \star_h u \wedge q := (-1)^{n-k} \int_T \langle u, q \rangle.$$

Appendix

This turns out to satisfy the expected integration by parts formula. That is, for all $u \in \mathcal{P}_r^- \Lambda^k(T)$ and $v \in \mathcal{P}_r^- \Lambda^{k-1}(T)$, we have the formula

$$\int_T \langle du, v \rangle = \int_T \langle u, \delta_h v \rangle;$$

where

$$\delta_h v := (-1)^{k-1} \star_h^{-1} d \star_h v.$$