# Differential Forms

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NA Reading group

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# Overview

We fix a smooth orientable Riemannian manifold $M$. That is, we have a locally Euclidean geometry with a pointwise inner product, which behaves smoothly. This structure lets us describe PDE on sub-manifolds, possibly of weaker regularity, in terms of

- differential $k$ forms;
- the exterior derivative;
- the Hodge star.

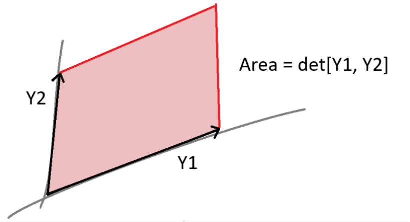
# Top-forms

A top-form $\mu$ encodes integration over $M$. It does this by assigning a scalar to each set of $n$ vectors $Y_1, \cdots, Y_n$. This yields a well-defined integral provided

$$\mu(AY_1, \cdots, AY_n) = \det(A)\, \mu(Y_1, \cdots, Y_n).$$

For example, in Euclidean space the volume form is given by

$$\mu_{\mathrm{Euc}}(Y_1, \cdots, Y_n) = \det[Y_1, \cdots, Y_n].$$



# $k$ forms

Similarly, we imagine a $k$ form $\alpha$ as something that assigns a scalar to each set of $k$ vectors $Y_1, \cdots, Y_k$. It should be alternating in the sense that for all permutations $\sigma$ we have

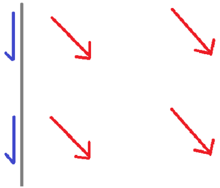
$$\alpha(Y_1, \cdots, Y_k) = \operatorname{sgn}(\sigma)\, \alpha(Y_{\sigma(1)}, \cdots, Y_{\sigma(k)}).$$

We denote the space of $k$ forms on $M$ by $\Lambda^k(M)$.

# Pull-back

A $k$ form encodes integration over $k$ dimensional sub-manifolds as follows: If $S$ is a sub-manifold of $M$, then a tangent vector in $S$ is also tangent to $M$. We say that the inclusion $i : S \to M$ induces a map $i^* : \Lambda^k(M) \to \Lambda^k(S)$ by

$$i^*\alpha(Y_1, \cdots, Y_k) := \alpha(i_* Y_1, \cdots, i_* Y_k).$$



# Wedge product

Given a $k$ form $\alpha$ and a $l$ form $\beta$, their wedge product is the $k+l$ form defined by

$$(\alpha \wedge \beta)(Y_1, \cdots, Y_{k+l}) = \frac{1}{k!l!} \sum_\sigma \mathrm{sgn}(\sigma) \alpha(Y_{\sigma(1)}, \cdots, Y_{\sigma(k)}) \beta(Y_{\sigma(k+1)}, \cdots, Y_{\sigma(k+l)}).$$

For example, if $dx$ and $dy$ are 1-forms then

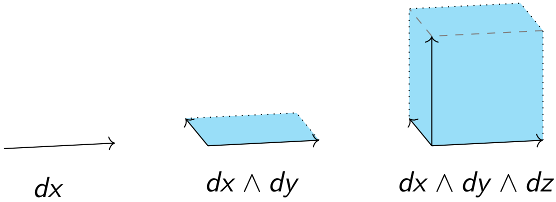
$$dx \wedge dy = dx \otimes dy - dy \otimes dx.$$

# Basis

If $dx_1, \cdots, dx_n$ is a basis for $\Lambda^1|_p$ then

$$\Lambda^k|_p = \mathrm{span}\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : i \in \mathcal{I}_k\},$$

where $\mathcal{I}_k$ is the set of multi-indices $1 \leq i_1 \leq \cdots \leq i_k \leq n$.



# Inner product

By duality we get an induced metric on $\Lambda^1(M)$. If $dx_1, \cdots, dx_n$ is the dual basis of $X_1, \cdots, X_n$ and we write $g_{ij} = \langle X_i, X_j \rangle$, then

$$\langle \sum_i a_i dx_i, \sum_j b_j dx_j \rangle = \sum_{i,j} a_i g^{ij} b_j.$$

This further extends to general $k$ forms by the Gram determinant

$$\langle dx_{i_1} \wedge \cdots \wedge dx_{i_k}, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \rangle = \det(\langle dx_{i_k}, dx_{j_l} \rangle)_{kl}.$$

# Volume form

The volume form is the positively oriented top-form $\mu_g$ which satisfies $\langle \mu_g, \mu_g \rangle = 1$. It provides a duality between 0-forms and top forms. This means we can integrate 0-forms

$$\int f := \int f \mu_g .$$

It also lets us compare the two different expressions we have for relating $\alpha \in \Lambda^k(M)$ to other forms

$$\begin{aligned} \alpha, \beta &\mapsto \langle \alpha, \beta \rangle \mu_g \quad & \beta \in \Lambda^k(M), \\ \alpha, \gamma &\mapsto \alpha \wedge \gamma \quad & \gamma \in \Lambda^{n-k}(M). \end{aligned}$$

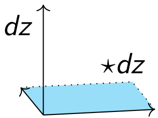
# Hodge star

The Hodge star $\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$ is uniquely defined by

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \mu_g.$$

It can be interpreted in terms of a duality

$$\begin{aligned} \text{span}\{dz\} &= \{v \in \mathbb{R}^n : v \perp \{dx, dy\}\}, \\ \text{span}\{dx, dy\} &= \{v \in \mathbb{R}^n : v \perp dz\}. \end{aligned}$$



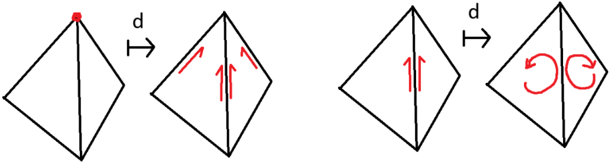
# Exterior derivative

The exterior derivative is adjoint to the boundary operator, in the sense that for all $\alpha \in \Lambda^{\dim T - 1}(T)$

$$\int_T d\alpha = \int_{\partial T} \alpha.$$

It can be defined in local coordinates by

$$d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$



# Exterior derivative

If $f \in \Lambda^0(M)$ then

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

If $\alpha \in \Lambda^k(M)$ and $\beta \in \Lambda^l(M)$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

# Co-differential

The co-differential is adjoint to the exterior derivative, in the sense that for all $\alpha \in \Lambda^{k-1}(T)$ and $\beta \in \Lambda^k(T)$

$$\int_T \langle d^\star \alpha, \beta \rangle = \int_T \langle \alpha, d\beta \rangle - \int_{\partial T} \alpha \wedge \star\beta.$$

It is given explicitly by $d^\star = (-1)^k \star^{-1} d \star$.

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# Common identities

In practice it is convenient to construct our differential forms inductively, and rely on identities instead of coordinates

- $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$;
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$;
- $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle$;
- $i_* d\alpha = d(i_* \alpha)$;
- ...

# Vector proxies

Given a basis $X_1, \cdots, X_n$ of $T_pM$ the musical isomorphism is defined by

$$\langle dx_i^\sharp, X_j \rangle = dx_i(X_j), \qquad X_i^\flat(X_j) = \langle X_i, X_j \rangle.$$

If $dx_1, \cdots, dx_n$ is the dual basis of $X_1, \cdots, X_n$ then

$$dx_i^\sharp = \sum_j g^{ij} X_j, \qquad X_i^\flat = \sum_j g_{ij} dx_j.$$

# Vector calculus of differential forms

Given a vector field $u$ let $\alpha = u^\flat$ and $\alpha' = \star u^\flat$. In $\mathbb{R}^3$ we have

$$\begin{array}{ccccc} \mathrm{div}(u) & \sim & d^\star \alpha & \sim & d\alpha' \\ \mathrm{curl}(u) & \sim & \star d\alpha & \sim & d\star\alpha'. \end{array}$$

We also get

$$\begin{aligned} \Delta u &= \mathrm{grad}\,\mathrm{div}(u) - \mathrm{curl}\,\mathrm{curl}(u) \\ &\sim (dd^\star + d^\star d)\alpha \\ &\sim (d^\star d + dd^\star)\alpha'. \end{aligned}$$

# Integration by parts

We get the divergence theorem

$$\int_T \mathrm{div}(u) = \int_{\partial T} u \cdot \nu \quad \Longleftrightarrow \quad \int_T d\alpha' = \int_{\partial T} \alpha'.$$

When $\dim M = 3$ and $\dim T = 2$ we recover the classic Stokes' theorem

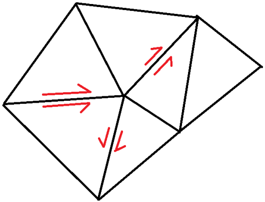
$$\int_T \mathrm{curl}(u) \cdot \nu = \int_{\partial T} u \quad \Longleftrightarrow \quad \int_T d\alpha = \int_{\partial T} \alpha.$$

# Standardized transformations

If $i : T \to T'$ and $\alpha$ is a top-form in $T'$ then

$$\int_T i_* \alpha = \int_{T'} \alpha.$$

The implication is that as long as we have single-valued pullbacks in our mesh, then we get a discrete Stokes' theorem for all $k$ forms.



# Examples

If $u$ satisfies

$$\begin{cases} \Delta u = f & \text{in } M \\ u \equiv 0, \ \operatorname{div}(u) \equiv 0 & \text{on } \partial M, \end{cases}$$

then $\alpha := u^\flat$ satisfies

$$\begin{cases} \Delta \alpha = f^\flat & \text{in } M \\ \alpha \equiv 0, \ d^\star \alpha \equiv 0 & \text{on } \partial M. \end{cases}$$

# Examples

If $u$ satisfies

$$\begin{cases} \Delta u = f & \text{in } M \\ u \cdot \nu \equiv 0, \ \operatorname{curl}(u) \equiv 0 & \text{on } \partial M, \end{cases}$$

then $\alpha := u^\flat$ satisfies

$$\begin{cases} \Delta \alpha = f^\flat & \text{in } M \\ \star\alpha \equiv 0, \ \star d\alpha \equiv 0 & \text{on } \partial M. \end{cases}$$

If we instead plug in $\alpha' := \star u^\flat$ we get the same equation as before!

# Summary

Advantages of differential forms:

- Handles higher dimensions and general metrics;
- less pitfalls;
- FEEC.

Advantages of vectors:

- More intuitive;
- particularly suited for the Euclidean setting.

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# Polynomial forms

We have a triangle $T$ with vertices $[0, \cdots, n]$, and with barycentric coordinates $\lambda_0, \cdots, \lambda_n$. Given a polynomial $q \in \mathbb{P}_r[\lambda_0, \cdots, \lambda_n]$, and a face $f$ with vertices $[f_0, \cdots, f_{k-1}]$, we get a form

$$q \, d\lambda_{f_0} \wedge \cdots \wedge d\lambda_{f_{k-1}}.$$

The span of all such forms is denoted by $\mathcal{P}_r\Lambda^k$.

# Polynomial forms

In this setting restrictions behave nicely. If $i : f \to T$ then

$$i_*(q \wedge q') = q|_f \wedge q'|_f, \quad i_*(dq) = dq|_f.$$

If $\lambda_i$ is linear, then we can compute $d\lambda_i$ by the fact that it is orthogonal to the face opposite the $i$-th vertex.

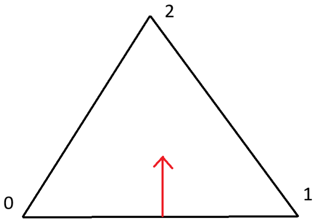


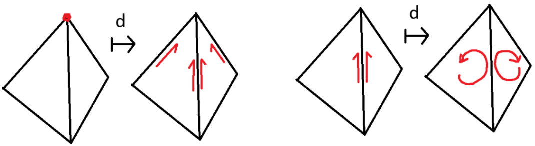

Figure: $d\lambda_2$

# Whitney forms

If $f$ is a face with vertices $[f_0, \cdots, f_k]$, it's corresponding Whitney form is defined as

$$\lambda_f := k! \sum_{i=0}^{k} (-1)^i \lambda_{f_i} d\lambda_{f_0} \wedge \cdots \wedge \widehat{d\lambda_{f_i}} \wedge \cdots \wedge d\lambda_{f_k}.$$

Assuming we take care of orientation, we get $\int_{f'} \lambda_f = \delta_{ff'}$ for all faces $f, f'$ of equal dimension. This realizes the discrete exterior derivative.



# Assembling the global spaces

We identify each vertex in the mesh with a scalar that is piecewise a barycentric coordinate. If these are continuous, then repeating the previous constructions yields spaces with single-valued pullback.

# A different example

If $X$ is a vector field such that $\mathrm{div}(X) \equiv c$, and $f$ is a scalar such that $df(X) = kf$, then

$$\int_T f = \frac{1}{k+c} \int_{\partial T} f \star X^\flat.$$

```
def star(v, x):
    r = x * x
    g11 = 1 - r[0, :]
    g12 = -x[0, :] * x[1, :]
    g22 = 1 - r[1, :]
    return np.array([
        -g12 * v[0, :] - g22 * v[1, :],
        g11 * v[0, :] + g12 * v[1, :]
    ]) / np.sqrt(1 - r[0, :] - r[1, :])
```

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# Overview

We recall the classic Poisson's equation: Find $u \in H^1(T)$ such that

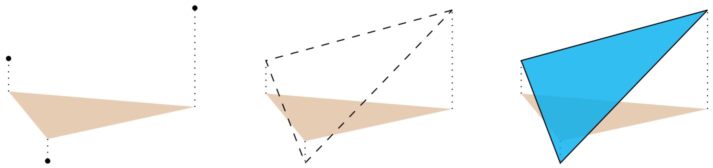
$$\begin{cases} \Delta u = f & \text{in } T \\ u|_{\partial T} = h. \end{cases}$$

The degrees of freedom are $f \in H^{-1}(T)$ and $h \in L^2(\partial T)$ (I think).

# Overview

The key idea is to iteratively construct the test functions:

$$\begin{cases} \Delta v \equiv 0 & \text{in } T \\ \Delta v|_e \equiv 0 & \text{for all edges } e \triangleleft T. \end{cases}$$



# Virtual $k$ forms

The corresponding Poisson equation on $k$ forms reads

$$\begin{cases} d^\star dv = f & \text{in } T, \\ d^\star v = g & \text{in } T, \\ v|_{\partial T} = h. \end{cases}$$

# Virtual $k$ forms

The corresponding Poisson equation on $k$ forms reads

$$\begin{cases} d^\star dv = f & \text{in } T, \\ d^\star v = g & \text{in } T, \\ v|_{\partial T} = h. \end{cases}$$

Repeating the previous construction now yields the Whitney forms

$$\mathfrak{W}^k = \{v \in \Lambda^k(T) : d^\star dv \equiv 0,\, d^\star v \equiv 0, \quad \forall T' \triangleleft T\}.$$

# Virtual $k$ forms

Suppose we have an exact sequence

$$0 \longrightarrow \mathbb{R} \hookrightarrow Z^n \xrightarrow{d^\star} \cdots \longrightarrow Z^0 \longrightarrow 0.$$

We then define our VE spaces by

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\ d^\star v \in d^\star Z^k,\ v|_{\partial T} \in V^k(\partial T)\}.$$

# Exactness

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\ d^\star v \in d^\star Z^k,\ v|_{\partial T} \in V^k(\partial T)\}.$$

Let $v \in V^k(T)$. Then

$$\begin{cases} d^\star d(dv) \equiv 0 \\ d^\star(dv) = d^\star dv \in d^\star Z^{k+1} \\ (dv)|_{\partial T} = dv|_{\partial T} \in V^{k+1}(\partial T) \quad \text{by induction.} \end{cases}$$

Therefore, $d$ maps $V^k(T) \to V^{k+1}(T)$.

# Exactness

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\ d^\star v \in d^\star Z^k,\ v|_{\partial T} \in V^k(\partial T)\}.$$

Suppose $v \in V^k(T)$ is such that $dv \equiv 0$, where $1 \leq k \leq n-1$. Then $dv|_{\partial T} \equiv 0$, so by induction there exists $u|_{\partial T} \in V^{k-1}(\partial T)$ such that $du|_{\partial T} = v|_{\partial T}$. Let $u \in V^{k-1}(T)$ such that

$$\begin{cases} d^\star du = d^\star v; \\ d^\star u \equiv 0; \\ du|_{\partial T} = v|_{\partial T}. \end{cases}$$

Then $v = du$. The base case of $k = \dim T$ is treated similarly, using Stokes to handle the extra dof that comes from integration.

# Exactness

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\ d^\star v \in d^\star Z^k,\ v|_{\partial T} \in V^k(\partial T)\}.$$

We have thus shown that the following sequence is exact

$$0 \longrightarrow \mathbb{R} \hookrightarrow V^0 \xrightarrow{d} \cdots \longrightarrow V^n \longrightarrow 0.$$

Note that this still works without our assumptions on $Z^k$.

# Degrees of Freedom

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\ d^\star v \in d^\star Z^k,\ v|_{\partial T} \in V^k(\partial T)\}.$$

The set of dof corresponding to

$$v \mapsto \int_T \langle v, z \rangle \qquad \forall z \in Z^k$$

is unisolvent on $V_0^k(T)$.

# Degrees of Freedom

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\, d^\star v \in d^\star Z^k,\, v|_{\partial T} \in V^k(\partial T)\}.$$

If $v \in V_0^k(T)$ is such that $\int \langle v, z \rangle = 0$ for all $z \in Z^k$ then

$$0 = \int_T \langle v, d^\star dv \rangle = \int_T \langle dv, dv \rangle - \underbrace{\int_{\partial T} v \wedge \star dv}_{=0},$$

and so $dv \equiv 0$.

# Degrees of Freedom

$$V^k(T) := \{v \in \Lambda^k : d^\star dv \in d^\star Z^{k+1},\ d^\star v \in d^\star Z^k,\ v|_{\partial T} \in V^k(\partial T)\}.$$

Then there exists $u \in V_0^{k-1}(T)$ such that $v = du$. We get that

$$0 = \int_T \langle du, z\rangle = \int_T \langle u, d^\star z\rangle - \underbrace{\int_{\partial T} u \wedge \star z}_{=0},$$

for all $z \in Z^k$. In particular $\int \langle u, d^\star du\rangle = 0$, and we conclude with the same argument as before that $v \equiv 0$.

# Summary

We have an exact sequence

$$0 \longrightarrow \mathbb{R} \hookrightarrow V^0 \xrightarrow{d} \cdots \longrightarrow V^n \longrightarrow 0.$$

The dof let us compute the $L^2$ projection of $V^k$ onto $Z^k$. Moreover, if $\zeta \in \Lambda^k$ is such that $d^\star d\zeta \in Z^k$, then we can almost compute

$$\int_T \langle dv, d\zeta \rangle = \int_{\partial T} v \wedge \star d\zeta - \int_T \langle v, d^\star d\zeta \rangle.$$

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# Contraction

A vector field $X$ contracts a $k$ form $\alpha$ to a $k-1$ form as follows:

$$X \lrcorner \alpha(Y_1, \cdots, Y_{k-1}) = \alpha(X, Y_1, \cdots, Y_k).$$

The contraction is related to the Hodge star by

$$X \lrcorner \alpha = (-1)^{n-k} \star (X^\flat \wedge \star^{-1} \alpha) \quad \forall \alpha \in \Lambda^k(M).$$

It also lets us express the Lie derivative as

$$\mathcal{L}_X \alpha = d(X \lrcorner \alpha) + X \lrcorner d\alpha.$$

# Boundary conditions

Given $i : \partial T \to T$ we have two equivalent ways of enforcing boundary conditions

$$\begin{aligned} \mathrm{tr}_{\partial T}(\nu^\flat \wedge \alpha) &\quad\leftrightarrow\quad i^*\alpha &\quad\leftrightarrow\quad \text{tt b.c.}, \\ \mathrm{tr}_{\partial T}(\nu \lrcorner \alpha) &\quad\leftrightarrow\quad i^*\star\alpha &\quad\leftrightarrow\quad \text{nn b.c.}, \end{aligned}$$

where $\nu$ is the normal vector field defined on $\partial T$.