Turbulent Flows

From Leray Conjecture to Kolmogorov Theory

Introduction to the numerical analysis of incompressible viscous flows by William Layton

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Archimedes of Syracuse (287–212 BC)

Hydrostatics

I. Newton (1642–1727)

$$F = ma$$

D. Bernoulli (1700–1782)

$$\frac{\rho v^2}{2} + p = \text{constant}$$

L. Euler (1707–1783)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla p}{\rho} + g$$

C.L.M.H. Navier (1785–1836) & G. Stokes (1819–1903)

$$\rho \frac{Du}{Dt} = -\nabla p + \nabla \cdot \tau + \rho f$$

J.C. Maxwell (1831–1879)

Boundary conditions

O. Reynolds (1842–1912)

Studied turbulence in experiments

H. Poincaré (1854–1912)

Method of sweeping — an early numerical approach

L.F. Richardson (1881–1953)

Energy cascade + eddy viscosity

J. Leray (1906–1998)

Weak solutions of Navier–Stokes + Leray conjecture

A.N. Kolmogorov (1903–1987)

K41: $\eta \sim L \, \mathrm{Re}^{-3/4}$

J. Smagorinsky (1924–2005)

Smagorinsky model (LES)

O.A. Ladyzhenskaya (1922–2004)

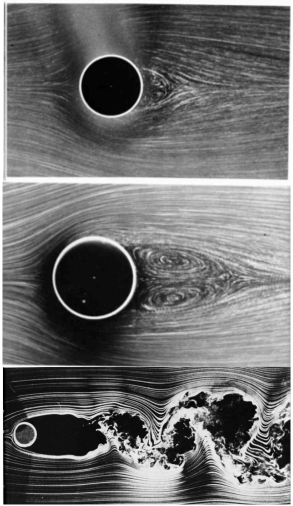
Rigorous analysis of the Navier–Stokes equations

Experiments from *An Album of Fluid Motion* by Milton Van Dyke:

Re $=$ 9.6

Re $=$ 26

Re $=$ 10000



Incompressible Navier-Stokes equations

Consider the fluid in a region $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$, bounded by a walls and driven by a body force $f(x,t)$,

$$\begin{aligned} u_t + u \cdot \nabla u - \mathrm{Re}^{-1} \Delta u + \nabla p = f, &\quad x \in \Omega, 0 < t \leq T, \\ \nabla \cdot u = 0, &\quad x \in \Omega, 0 < t \leq T, \\ u(x,0) = u_0(x), &\quad x \in \Omega, \\ u|_{\partial\Omega} = 0, &\quad 0 < t \leq T, \\ \int_\Omega p \, \mathrm{d}x = 0, &\quad 0 < t \leq T. \end{aligned}$$

Goal:

- Know the Leray theory.
- Understand Richardson's **qualitative** description of energy cascade.
- Understand Kolmogorov's **quantitative** description of the energy cascade, direct numerical simulation (DNS).
- Understand why we need turbulence models, large eddy simulation (LES).
- Current sheet and MHD turbulence and many open questions.

Section 1

The Leray theory

Consider solving

$$u'(t) = F(t, u(t)), \quad t \in I \subset \mathbb{R}, \quad u(0) = u_0.$$

Well-posedness (Hadamard 1902)

- existence
- uniqueness
- regularity: the solution depends continuously on the data

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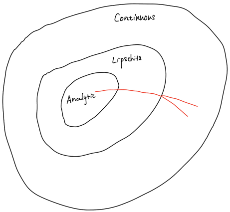
Local existence and uniqueness:

- Cauchy-Kovalevskaya: F is analytic.
- Cauchy-Lipschitz: F is Lipschitz.

Local existence ~~and uniqueness~~:

- Cauchy-Peano: F is continuous.



Solving a PDE in a modern way

- $\times$Find a explicit expression.
- $\checkmark$Find a function space large enough such as our solution **exists**, at the same time we want our function space small enough such that our solution is **unique**.

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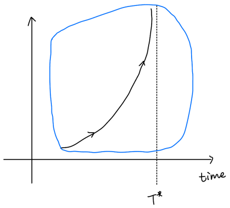
Local and global solution

- The solution is continued in I.
- The solution blows up at T^*, e.g. $u' = u^2, u(0) = u_0 > 0$, the solution is $u(t) = \frac{u_0}{1-u_0 t}$.



- The notion of the weak solution is due to Leray, who called it **turbulent solutions**.

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$$\frac{1}{2}\|u(t)\|^2 + \mathrm{Re}^{-1}\int_0^t \|\nabla u(t')\|^2 dt' = \frac{1}{2}\|u_0\|^2 + \int_0^t (f(t'), u(t'))dt'.$$

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- Leray does care about the physics — the weak solution is not weak.

Weak solution

Let $u_0 \in H$, $f \in L^2(0,T;V')$. The velocity field u is said to be a weak solution of NSE when

- $u \in L^2(0,T;V) \cap L^\infty(0,T,H)$, and
- u satisfies the integral relation

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle_{V',V} + \int_\Omega u \cdot \nabla u \cdot v + \mathrm{Re}^{-1} \int_\Omega \nabla u \cdot \nabla v = \langle f, v \rangle_{V',V}$$

  in the sense of distributions in time, for all $v \in V$ and

$$u(0,\cdot) = u_0.$$

where

$$V = \{v \in \mathbb{H}_0^1(\Omega), \nabla \cdot v = 0\},$$
$$H = \{v \in \mathbb{L}^2(\Omega), \nabla \cdot v = 0, v \cdot n|_{\partial\Omega} = 0\}.$$

Some results

- In 2D, the weak solution exists and is unique.
- In 3D, the the weak solutions exist, **but uniqueness is unknown**.

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- Paraphrase: Turbulence is assciated with a possible (yet unproved) **breakdown of the uniqueness** of weak solutions to the NSE.
- In the Clay Mathematical Conference, Oxford 2025, Thomas (Yizhao) Hou presented the latest progress

Hou, Thomas, Yixuan Wang, and Changhe Yang. "Nonuniqueness of Leray-Hopf solutions to the unforced incompressible 3D Navier-Stokes Equation." arXiv preprint arXiv:2509.25116 (2025).

Section 2

1922: Richardson's energy cascade

Big whirls have little whirls
That feed on their velocity,
And little whirls have lesser whirls,
And so on to viscosity.

— L. F. Richardson, *Weather Prediction by Numerical Process* (1922)

For high Re problems with no external force f:

$$\frac{d}{dt}\mathcal{E} = -\,\mathrm{Re}^{-1}\,\|\nabla u\|^2$$

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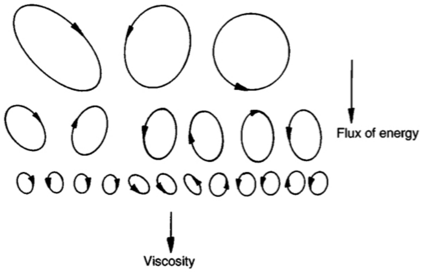

(Davidson 2017)

Section 3

1941: Kolmogorov theory

Decomposition into wave number/length scales $k = \frac{1}{L}$

- Time averaged energy dissipation rate $\langle \varepsilon \rangle = \langle \frac{d}{dt} E \rangle \sim \frac{\text{Length}^2}{\text{Time}^3}$.
- Averaged kinetic energy across wave number $E(k) = \frac{L}{2\pi} \sum \frac{1}{2} |\hat{u}(k,t)|^2 \sim \frac{\text{Length}^3}{\text{Time}^2}$.

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K41 theory : homogeneous, isotropic turbulence

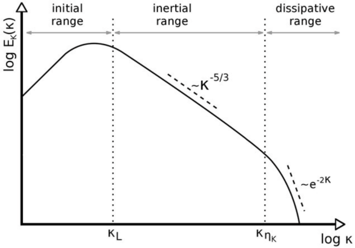
At high enough Reynolds numbers there is a range of wave numbers

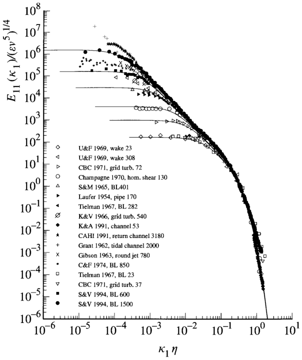
$$0 < k \leq C(L\,\mathrm{Re}^{-\frac{3}{4}})^{-1},$$

known as **the inertial range**. In this range,

$$E(k) = \alpha \langle \varepsilon \rangle^{\frac{2}{3}} k^{-\frac{5}{3}}.$$

- L is the large length scale.
- α is called the universal Kolmogorov constant $(1.4 \sim 1.7)$.
- $\langle \varepsilon \rangle$ depends on the flow.





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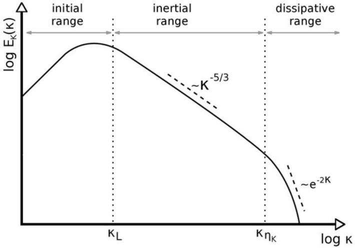

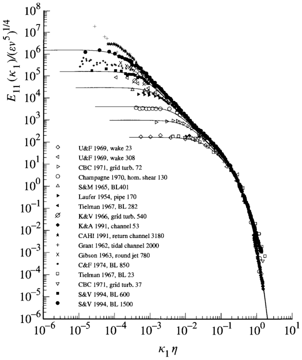

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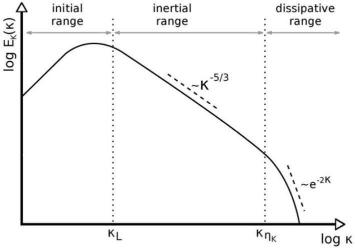

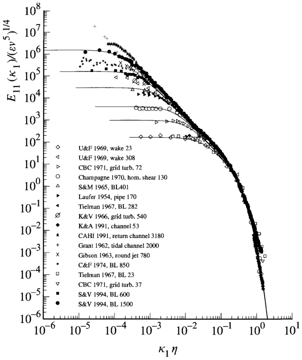

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- A power relation is observed when $k_1\eta < 0.1$, and the slope matches the number $-\frac{5}{3}$!

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- Insert
$$\frac{\text{length}^3}{\text{time}^2} = \text{length}^{2a-b}\text{time}^{-3a} \implies 3a = 2, 2a - b = 3 \implies a = \frac{2}{3}, b = -\frac{5}{3}$$

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Length	L	η
Reynolds number	$\mathrm{Re} = \dfrac{UL}{\nu}$	$\mathrm{Re}_{\mathsf{small}} = \dfrac{v_{\mathsf{small}}\eta}{\nu}$

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- Equate the large scale and the small scale

$$\boxed{\nu\left(\frac{v_{small}}{\eta}\right)^2 \simeq \frac{U^3}{L}}$$

Solve for

$$\boxed{\frac{v_{small}\eta}{\nu} \simeq 1.} \implies v_{small} \simeq \frac{\nu}{\eta}$$

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Kolmogorov scale tells us that the smallest turbulent eddies shrink like $\mathrm{Re}^{-\frac{3}{4}}$.

For 3D flow simulation, the persistent eddies require taking

$$\Delta x = \Delta y = \Delta z = O(\mathrm{Re}^{-\frac{3}{4}}).$$

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Small airplane	2×10^7	$10^{-13}\ m$	10^{16}
Atmospheric flows	Big!!	$1\ mm$	10^{20} and higher.

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- Large eddy simulation (LES): select a length scale δ of interest, simulate eddies of size $l \geq O(\delta)$.

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which yields $O(Re^{\frac{9}{4}})$ mesh points.

Flow	Re	$l = O(\mathrm{Re}^{-3/4})$	# Mesh points
Model airplane	7×10^4	$10^{-3}\ m$	10^{10}
Subcompact car	6×10^5	$10^{-8}\ m$	10^{12}
Small airplane	2×10^7	$10^{-13}\ m$	10^{16}
Atmospheric flows	Big!!	$1\ mm$	10^{20} and higher.

- The above procedure is called direct numerical simulation (DNS) (Very expensive).
- Large eddy simulation (LES): select a length scale δ of interest, simulate eddies of size $l \geq O(\delta)$.
- **How to choose such a length scale δ?**

Section 4

Turbulence model (LES)

Yon foaming flood seems motionless as ice
Its dizzy turbulence eludes the eye
Frozen by distance
– W. Wordsworth, 1770–1850, "Address to Kilchurn Castle"

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Smoother $g_\delta(x) := \delta^{-d} g\left(\frac{x}{\delta}\right)$,

Smoothing $\overline{u}(x) = (g_\delta * u)(x) = \int_{\mathbb{R}^d} g_\delta(x-y)u(y)dy$,

Fluctuation $u' = u - \overline{u}$

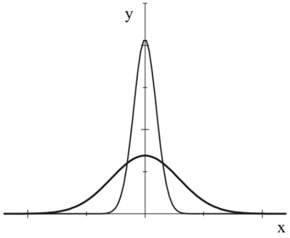


Figure 10.1. *A Gaussian filter (heavy) and rescaled (thin).*

The smoother $g_\delta(x)$.

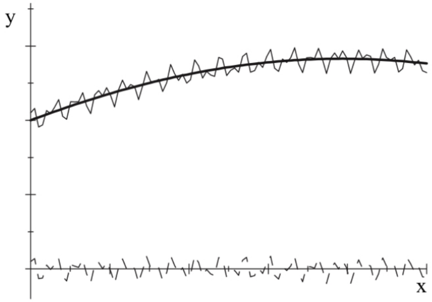
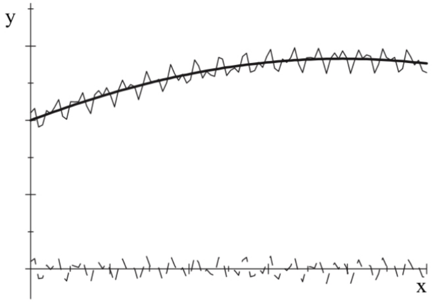


Figure 10.2. *A curve, its mean (heavy line) and fluctuation (dashed).*



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- The smoother g_δ will eliminate any local oscillations smaller than $O(\delta)$.

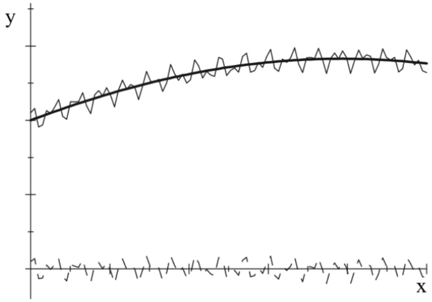

Figure 10.2. *A curve, its mean (heavy line) and fluctuation (dashed).*

- The smoother g_δ will eliminate any local oscillations smaller than $O(\delta)$.
- Therefore, we can apply this convolution (filter) to NSE

$$g_\delta * NSE(u) = g_\delta * f$$

SFNES = spaced-filtered-NSE

$$\overline{u}_t + \nabla \cdot (\overline{u}\,\overline{u}) + \nabla \overline{p} - Re^{-1} \Delta \overline{u} + \nabla \cdot \mathbf{R}(u,u) = \overline{f},$$
$$\nabla \cdot \overline{u} = 0$$

where

$$\mathbf{R}(u,u) := \overline{uu} - \overline{u}\,\overline{u}$$

is called subfilter scale stress tensor.

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- The additional stress term $\nabla \cdot R(u,u)$ represents the unresolved scaled exerts upon the resolved scales. the
- Other models like NSE-α model, using

  $$\overline{u} = (1 - \alpha^2\Delta)^{-1}u.$$

  as a filter.

Section 5

Current sheet and MHD turbulence

1D magnetic relaxation (Bajer and Moffatt 2013) solved by finite difference method.

$$\frac{\partial B}{\partial t} = \frac{1}{S^2}\frac{\partial^2 B}{\partial x^2} - \frac{\partial}{\partial x}(uB), \quad \frac{\partial u}{\partial x} = \frac{1}{2}[B^2(x,t) - \langle B^2(x,t)\rangle], \quad \Omega = [0, 2L].$$

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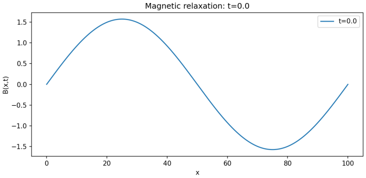
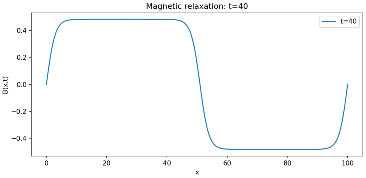


(a) $t = 0$



(b) $t = 40$

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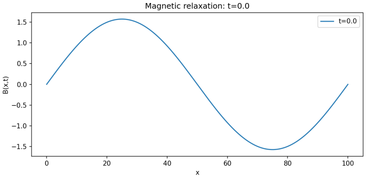

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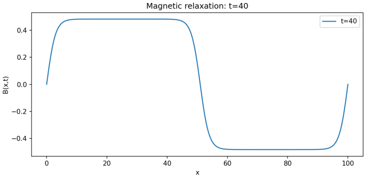

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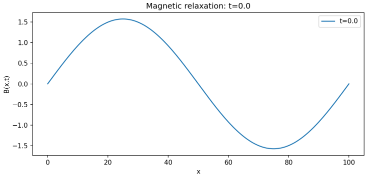

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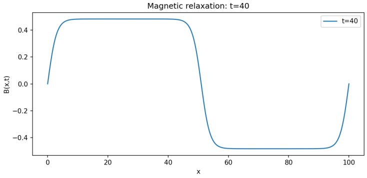

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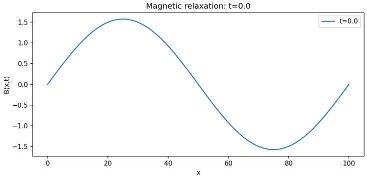

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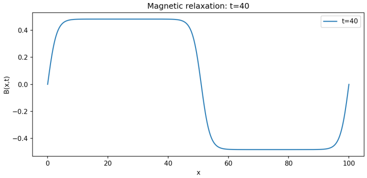

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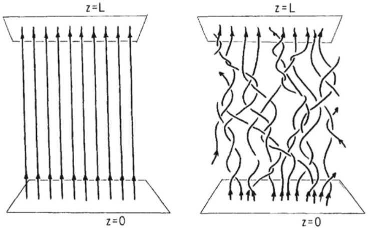
- A fundamental question in plasma physics: given an initial data, what does the system evolve to?
- A discontinuous solution (current sheet) is formed at $t = 40$.
- **The initial data is smooth, but a discontinuous solution is formed. Is this a general phenomenon for MHD?**

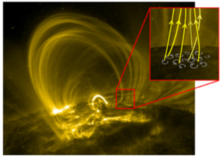
The Parker conjecture (1972)

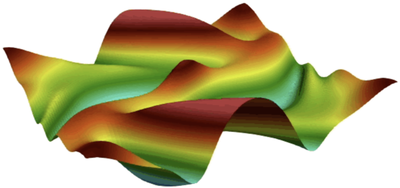
For almost all possible flows, the magnetic field develops current sheet (**tangential discontinuity**) during ideal magnetic relaxation to a force-free equilibrium.



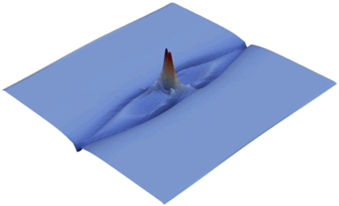
(Pontin and Hornig 2020).

Eugene N. Parker

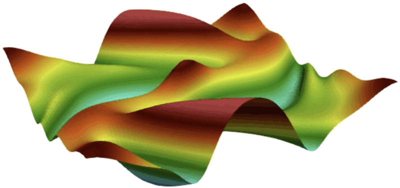




(a) Orszag-Tang vortex.



(b) Island coalescence in Tokamak.



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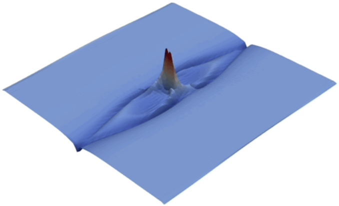

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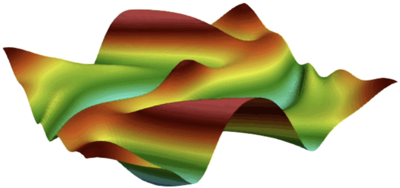

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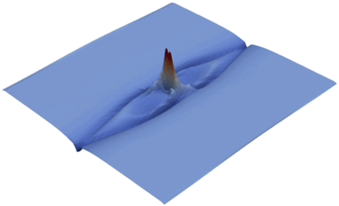

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- The current sheet has an associated singular current K

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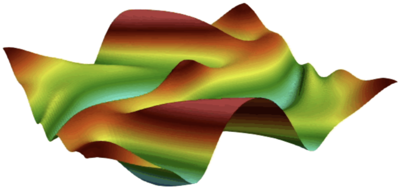

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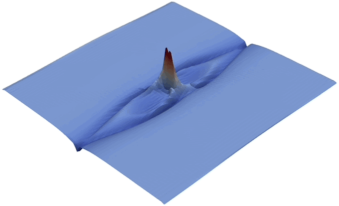

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Why do we care about the current sheet?

The plasma turbulence is the consequence of and not the reason for the current sheet appearance. (Vainshtein, Bykov, and Toptygin 1989).

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- **Discontinuous solutions** of ideal MHD (Landau and Lífshíts 1884):

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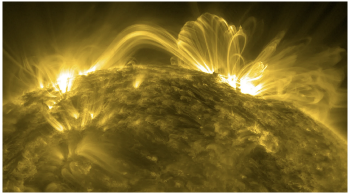

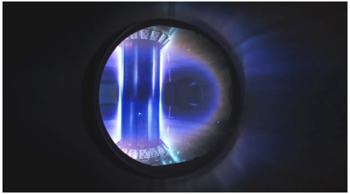

- Generalized Orszag-Tang vortex on $\Omega = [0, 2\pi]^2$, with initial conditions

$$u_0 = \hat{z} \times \nabla\phi = (-\partial_y\phi, \partial_x\phi), \quad B_0 = \hat{z} \times \nabla\psi = (-\partial_y\psi, \partial_x\psi),$$

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$$\phi(x, y) = \cos(x + 1.4) + \cos(y + 0.5), \quad \psi(x, y) = \cos(2x + 2.3) + \cos(y + 4.1).$$

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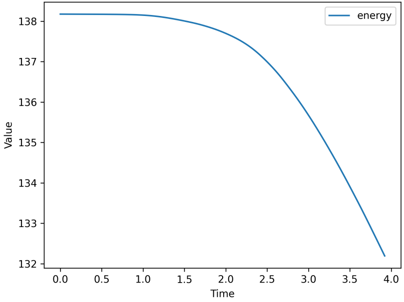
(a) $t = 0.02$ (b) $t = 0.7$ (c) $t = 1.0$ (d) $t = 2.0$

Plot of current $j = \nabla \times B$: $\Delta t = 0.02$, $\mathrm{Re} = \mathrm{Re}_m = 10^5$.

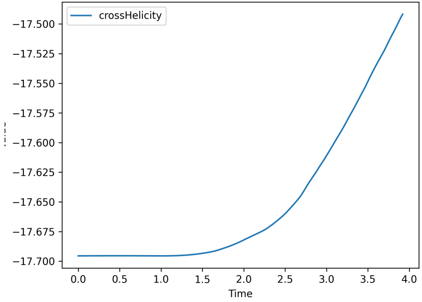
Conservation laws (with nice BCs)

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\int \frac{1}{2}u^2 + B^2 dx = -\mathrm{Re}_m^{-1}\int j^2 dx - \mathrm{Re}^{-1}\int w^2 dx,$$
$$\frac{d}{dt}\mathcal{H}_c = \frac{d}{dt}\int u\cdot B dx = -(\mathrm{Re}^{-1} + \mathrm{Re}_m^{-1})\int j\cdot w dx,$$
$$\frac{d}{dt}\mathcal{H}_m = \frac{d}{dt}\int A\cdot B dx = -\mathrm{Re}_m^{-1}\int j\cdot B dx.$$

- The total energy, cross helicity are conserve in the ideal limit $\mathrm{Re} = \mathrm{Re}_m = \infty$.
- In 2D, the magnetic helicity is trivially zero.

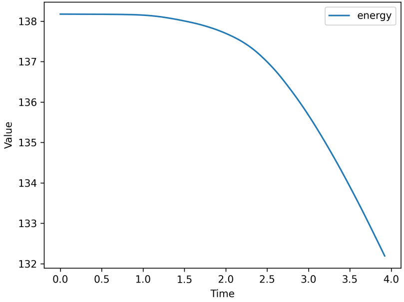


(a) Total Energy: $\mathcal{E} = \frac{1}{2}(\|u\|^2 + c\|B\|^2)$



(b) Cross helicity $\mathcal{H}_c = \int u \cdot B dx$

- There is a clear boundary of the dynamics.



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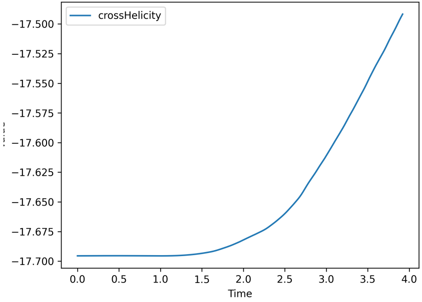

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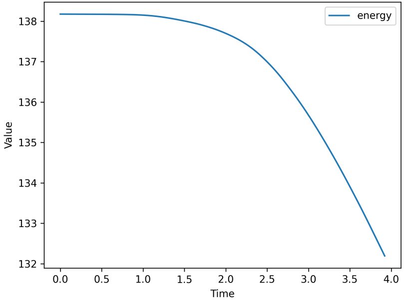

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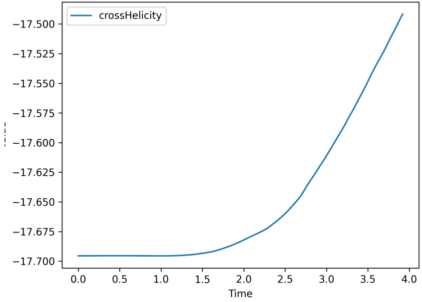

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- There is a clear boundary of the dynamics.
- Stage 1: $t = [0, 1.0]$: Large-scale conservation of energy and cross helicity.
- Stage 2: $t = (1.0, 4.0]$. Small-scale dissipation of energy and cross helicity.

After $t = 1.0$ (my simulation becomes rubbish), some mysterious things happen – the current sheet might remain regular/break...who knows?

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Long term behaviours of my solution

- Some dynamical variable, e.g. the vorticity $w = \nabla \times u$ or current $j = \nabla \times B$ blows up i.e. becomes infinite in a finite time, called **finite-time singularity (FTS)**.
- The growth is only exponential, such that a singularity is reached only after an infinite period, i.e. the solution remains **regular**.

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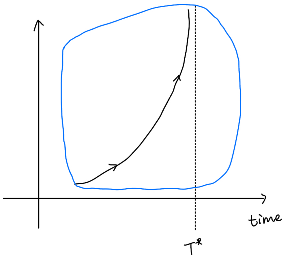

An useful criteria in the search of singularity comes from the Beale-Kato-Majda theorem (BKM):

BKM (Beale, Kato, and Majda 1984)

If the smooth initial data of an ideal fluid leads to singularity at time $t = T^*$, then

$$\int_0^{T^*} \|w(\cdot, t)\|_\infty dt = \infty.$$

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If a power-law divergence at T^* is assumed,

$$\|w(\cdot, t)\|_\infty = C[T^* - t]^{-\beta}, \text{ for } t \to T^*.$$

The flow has a finite-time singularity at $t = T^* \iff \beta \geq 1$.

BKM in MHD (Caflisch, Klapper, and Steele 1997)

If the smooth initial data of an ideal MHD leads to singularity at time $t = T^*$, then

$$\int_0^{T^*} \|w(\cdot,t)\|_\infty + \|j(\cdot,t)\|_\infty dt = \infty.$$

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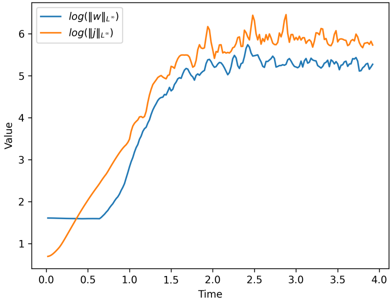
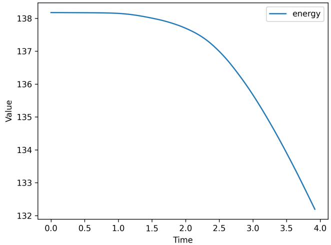
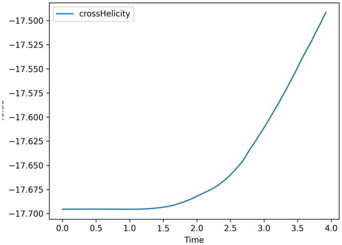


- Plot of $\log(\|w\|_\infty)$ and $\log(\|j\|_\infty)$.
- **A linear relationship** implies exponential growth of $\|j\|_\infty$, i.e.

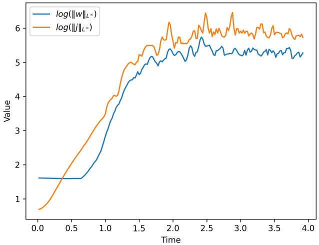
  $\|j\|_\infty \sim \alpha \exp(\beta t)$, where β is the slope.



(a) Energy



(b) Cross helicity

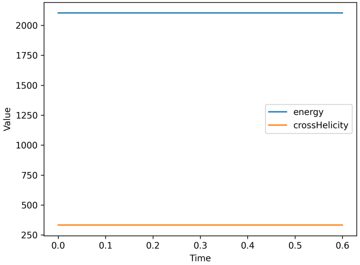


(c) BKM

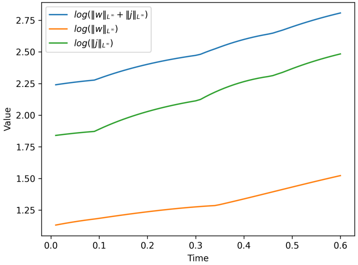
Conclusion:

- On the large scale, the diffusion is negligible, **the energy and cross helicity are conserved**.
- The current sheet becomes thinner and thinner and its magnititude **growths exponentially** until it reaches some scale δ, when diffusion can no longer be neglected.
- On the small scale, **the energy and the cross helicity are not conserved** due to the diffusion effects.

- Generalized Orszag-Tang vortex on $\Omega = [0, 2\pi]^3$



(a) Evolution of the energy and cross helicity for ideal case $\mathrm{Re} = \mathrm{Re}_m = \infty$.



(b) $\log(\|j\|_\infty)$, $\log(\|w\|_\infty)$ and $\log(\|w\|_\infty + \|j\|_\infty)$

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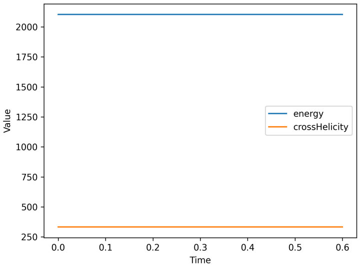

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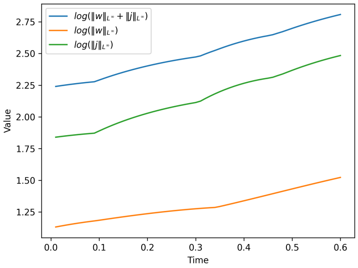

(b) $\log(\|j\|_\infty)$, $\log(\|w\|_\infty)$ and $\log(\|w\|_\infty + \|j\|_\infty)$

- My mesh is $4 \times 4 \times 4$, what happens if we increase the resolution?

A large-scale computation (but not structure-preserving) was done 25 years ago:

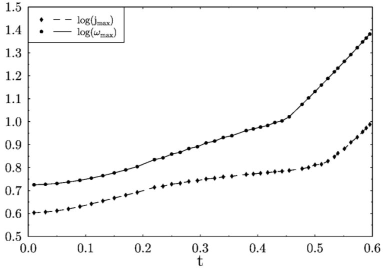

FIG. 3. Logarithm of L^∞-norms of current density and vorticity over time.

4096^3 + adaptive mesh refinement (Grauer and Marliani 2000)

Their conclusion

The ideal MHD does not have a finite-time singularity! The current sheet growths exponentially but will remain regular in finite time.

Section 7

Questions

Some numerical evidence of open problems: **long term behaviours of the solutions**

Incompressible ideal fluids

There exists a finite-time singularity of vorticity (Kerr 1993), (Pelz and Gulak 1997), (Grauer, Marliani, and Germaschewski 1998).

Incompressible ideal MHD

No finite-time singularity in 2D MHD (Biskamp and Welter 1989).
No finite-time singularity in 3D MHD (Politano, Pouquet, and Sulem 1995), (Grauer and Marliani 2000).

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  "In MHD, the Onsager principle of minimum energy dissipation **may be replaced by magnetic helicity conservation**, as suggested by the Taylor conjecture, which changes the system's effective dimensionality."
- **This statement highlights the significant role of the ideal invariants!**

- The conservation/dissipation laws ($\mathcal{E}$, $\mathcal{H}_m$, $\mathcal{H}_c$) govern the decay of MHD turbulence – called selective decay.

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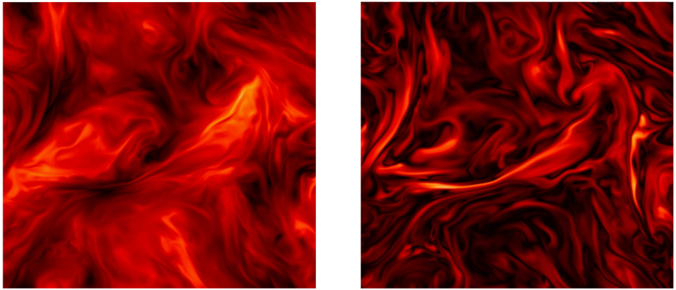
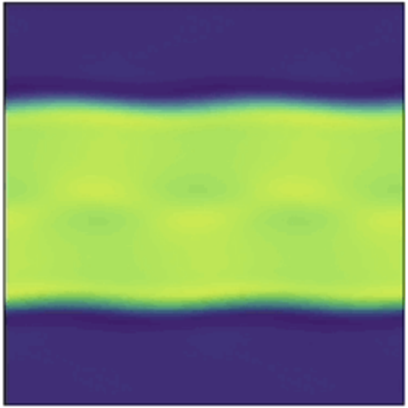
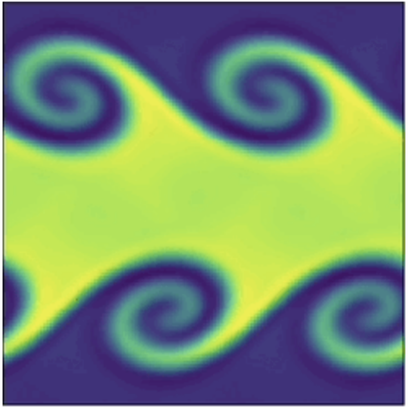


(a) $|\boldsymbol{u}|$ (b) $|\boldsymbol{B}|$

Figure from (Schekochihin 2022)



(a) $t = 0$



(b) $t = T$

Kelvin-Helmholtz instability, **spontaneous large scale behaviours** are observed.

Can we get the same conclusion if we use structure-preserving scheme to reproduce those experiments?

Our structure-preserving schemes yield more trustable results, and our conclusion will be stronger...

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How to detect singular structures in numerical simulations?

BKM exponents estimate, energy spectrum estimate, posterior estimates [Kaibo], timestepping technique for blow up...also, the high order method [Boris, Ganghui] will be very important!

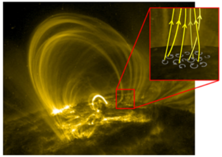
The Parker conjecture (1972)

For almost all possible flows, the magnetic field develops current sheet (**tangential discontinuity**) during ideal magnetic relaxation to a force-free equilibrium.

Numerical approaches

- **Helicity-preservation**: He, M., Farrell, P. E., Hu, K., & Andrews, B. (2025). Helicity-preserving finite element discretization for magnetic relaxation. SIAM Journal on Scientific Computing.
- **Boundary conditions**: periodic, magnetically closed, line-tied.
- **Representation of the current sheet by FEEC**, ongoing work with Kaibo.

Eugene N. Parker



Summary

- Leray's turbulent solution (weak solution).
- Richardons' energy cascade: energy transition from large-scale (input) to the small scale (output).
- Kolmogorv scale – The expensive computation in DNS.
- Spacial avaraging of turbulence model (LES).
- The current sheet (structure, formation, destruction) is essential to understand the MHD turbulence.
- Discussion: the role of the structure-preservating schemes (FEEC) in turbulence modeling and many open questions in hydrodynamics and MHD.

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Thank you!

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