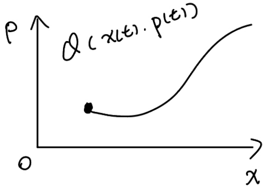
Hidden structures of dissipative equations leading to equilibria

Mingdong He[1]

[1]Mathematical Institute, University of Oxford

Phase Space

- Consider a particle moving along a line



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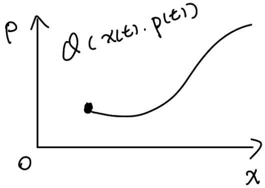

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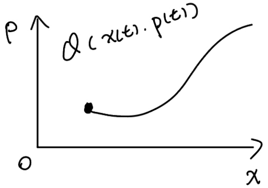

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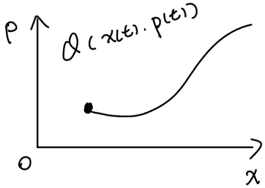

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$$H(x, p) = \frac{p^2}{2m} + V(x)$$

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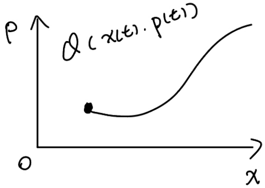

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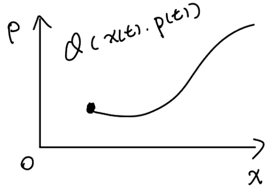

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Hamilton's Equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

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What if I have a quantity $Q(x(t), p(t))$? How does it change w.r.t time?

For a general quantity $Q(x(t), p(t))$

$$
\begin{aligned}
\frac{d}{dt}Q(x(t), p(t)) &= \frac{\partial Q}{\partial x}\frac{dx}{dt} + \frac{\partial Q}{\partial p}\frac{dp}{dt} \\
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The phase space does not need to be parametrized by t, it can be any parameter

$$\frac{dQ}{d\lambda} = \{Q, G\}$$

where G is called a **generator**.

We can choose the generator as $G = p$:

$$\frac{dx}{d\lambda} = \{x, p\} = \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial x}{\partial p}\frac{\partial p}{\partial x} = 1,$$
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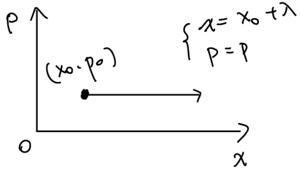

Space translation

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Noether's theorem

Conservation $\iff$ symmetry

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- **Casimir**: due to the kernel of the Poisson bracket.

$$\frac{d}{dt}C = \{C, H\} = 0, \qquad \forall H.$$

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Introducing scalar stream function ψ

$$u = \nabla^{\perp}\psi = (\partial_y \psi, -\partial_x \psi), \qquad w = \nabla \times \psi = \partial_x(-\partial_y \psi) - \partial_y(\partial_x \psi) = -\Delta\psi.$$

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Rewrite the advection term

$$u \cdot \nabla w = (\partial_y \psi)\partial_x w - (\partial_x \psi)\partial_y w = -[w, \psi]$$

where the Jacobian bracket is defined as

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The 2D Euler equation can be written as

$$\partial_t w = [\psi, w]$$

- Bracket for functional: Lie-Poisson bracket

$$\frac{df}{dt}(w) = \{f, H\}(w) = \left\langle w, \left[\frac{\delta f}{\delta w}, \frac{\delta H}{\delta w}\right]\right\rangle$$

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- We then recover the 2D Euler equation!

Energy (antisymmetry of the Poisson bracket)

The kinetic energy can be written in terms of ψ and ω:

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- We used $\langle a, [b, c]\rangle = \langle b, [c, a]\rangle = \langle c, [a, b]\rangle$
- This is a **dynamical invariant**, tied to the Hamiltonian structure.

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since

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- Example: $\Phi(s) = \frac{1}{2}s^2$ gives the enstrophy $(\Phi(s) = s^p, p = 3, 4 ...)$

Section 3

Mathematical challenges: computation of equilibria

Equilibria of Euler equation

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Equilibria of ideal MHD equation

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A great idea: **can we modify the Hamiltonian system to an artificial dynamical system that relaxes to an equilibrium of the considered physical system?**

- Bressan, C., Kraus, M., Maj, O., & Morrison, P. J. (2025). Metriplectic relaxation to equilibria. arXiv preprint arXiv:2506.09787.

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Metriplectic dynamics (artificial relaxation): u_∞

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Variational principle (physical constraint): u_*

$$\min\{S(u) : u \in V, \quad H(u) = H(u_0)\}$$

Section 4

Steady state via variational principle

Minimization

$$\min\{S(u) : u \in V, \quad H(u) = H(u_0)\} \implies \boxed{\frac{\delta S}{\delta u}(u_e) = \lambda \frac{\delta H}{\delta u}(u_e)}$$

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This is due to the constrained minimization with a functional $\mathcal{J}[u] = S(u) - \lambda(H(u) - H_0)$

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Applications:

- 2D steady Euler:

$$\mathcal{J}[\psi] = \frac{1}{2}\int_\Omega |\nabla\psi|^2 - \lambda\psi^2 dx \implies \Delta\psi = -\lambda\psi.$$

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$$\mathcal{J}[u] = \frac{1}{2}\int_\Omega \frac{u^2(R,z)}{CR^2 + D}\frac{dRdz}{R} - \lambda u(R,z)\psi(R,z)\frac{dRdz}{R} \implies -\Delta^*\psi = \lambda(CR^2 + D)\psi.$$

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- 3D Force-free field:

$$\mathcal{J}[u] = \frac{1}{2}\int_\Omega |B|^2 - \lambda B \cdot A dx \implies \nabla \times B = \lambda B.$$

Section 5

Metriplectic dynamics: artificial relaxation

A modified Hamiltonian system

$$\frac{d}{dt}F(u) = \{F(u), H(u)\} + (F(u), S(u))$$

- $\{\cdot, \cdot\}$: Poisson bracket (anti-symmetric).
- $(\cdot, \cdot)$: metric bracket (symmetric, negative semi-definite).

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- The entropy is dissipated at constant Hamiltonian.
- If S is bounded below, the system will evolve on the manifold $H(u) = H(u_0)$, toward a state that satisfies $(S, S) = 0$, i.e.

$$\{F, H\}(u_e) + (F, S)(u_e) = 0.$$

We can define the bracket

$$\{A, B\} = \int_\Omega \frac{\delta A}{\delta u}(x) J(u) \frac{\delta B}{\delta u}(x) d\mu(x),$$

$$(A, B) = -\int_\Omega \frac{\delta A}{\delta u}(x) K(u) \frac{\delta B}{\delta u}(x) d\mu(x)$$

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Strong form of the dynamics

$$\frac{d}{dt} F(u) = \{F, H\} + (F, S) \implies \boxed{\frac{d}{dt} u = J(u) \frac{\delta H}{\delta u} - K(u) \frac{\delta S}{\delta u}}.$$

Steady state

If u_e is the steady state of the variational principle, then it is also the steady state of the metriplectic dynamics. The converse is not true.

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Proof: since u_e statisfies $\frac{\delta S}{\delta u} = \lambda \frac{\delta H}{\delta u}$, then

$$\begin{aligned}\frac{d}{dt}u &= J(u)\frac{\delta H}{\delta u} - K(u)\frac{\delta S}{\delta u}|_{u_e}\\ &= (J(u) - \lambda K(u))\frac{\delta H}{\delta u}\\ &= 0,\end{aligned}$$

due to the compatibility condition

$$\begin{aligned}\{F, S\} = 0 &\implies J(u)\frac{\delta S}{\delta u} = 0,\\ (F, H) = 0 &\implies K(u)\frac{\delta H}{\delta u} = 0.\end{aligned}$$

Therefore,

$$\boxed{u^*_{\text{variational}} \subset u^*_{\text{dynamics}}}$$

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Completely relaxed state

If the solution of the metriplectic dynamics satisfies

$$\lim_{t\to\infty} u(t) = u^*_{\mathsf{variational}}, \qquad H(u^*_{\mathsf{variational}}) = H_0.$$

We say the system is **completely relaxed**.

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- It depends on the null space of the metric brackets.

Section 6

How to construct the metric bracket?

- Collision-like brackets: computationally expensive due to nonlocality.

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- Diffusion-like brackets: computationally friendly.

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Diffusion-like bracket

The diffusion bracket is defined as

$$(F, G) = -\int_{\Omega} L_i \left(\frac{\delta F}{\delta u}\right) D_{ij} L_i \left(\frac{\delta G}{\delta u}\right) d\mu(x)$$

where

- L_i: $V \to V'$ is some linear operator.
- $H = H(u)$ and $\frac{\delta H}{\delta u} \in V$.
- $D = D_{ij}(x)$ s.t. it is
  - symmetric, positive semi-definite
  - $g_i D_{ij} = 0$ where $g_i = L_i\left(\frac{\delta H}{\delta u}\right)$.
- μ: a positive measure on Ω.

Properties of diffusion-like bracket

- $(F, G) = (G, F)$.
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Two diffusion-like brackets: div-grad bracket and curl-curl-like bracket.

curl-curl bracket $L = \nabla\times$

$$(F, S) = -\int_\Omega \left(\nabla \times \frac{\delta F}{\delta u}\right) \cdot D\left(\nabla \times \frac{\delta S}{\delta u}\right) dx = -\int_\Omega \frac{\delta F}{\delta u} \nabla \times \left[D\left(\nabla \times \frac{\delta S}{\delta u}\right)\right] dx$$

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We end up with

$$\boxed{\frac{\partial B}{\partial t} = \nabla \times \left(B \times (B \times (\nabla \times B))\right)}$$

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$$u = (\nabla \times B) \times B.$$

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- Magneto-frictional equation.
- Dissipative entropy (magnetic energy): $\frac{d}{dt} \int B \cdot B \,\mathrm{d}x = -\|(\nabla \times B) \times B\|^2$.
- Conserved Hamiltonian (magnetic helicity): $\frac{d}{dt} \int A \cdot B \,\mathrm{d}x = 0$.

Selective decay (non-ideal)

- In 3D fluid or MHD, energy and helicity decay at different rates.
- One quantity is dissipative while another one is approximately constant.
- Statistical mechanics predictions: large-scale behaviour should be presented instead of random small scale behaviour.
- Numerical experiments: small-scale behaviour.

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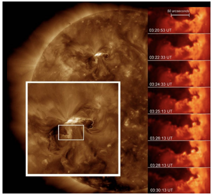

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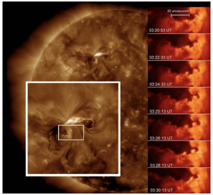

Question

We know that Lie-Poisson bracket yield two types of conservation, can we modify to impose such selective decay?

Again, consider Hamiltonian H and Casimir C:

$$\frac{df(w)}{dt} = \underbrace{\left\langle w, \left[\frac{\delta f}{\delta w}, \frac{\delta H}{\delta w}\right]\right\rangle}_{\{f,H\}} - \theta \left\langle \left[\frac{\delta f}{\delta w}, \frac{\delta H}{\delta w}\right], L\left[\frac{\delta C}{\delta w}, \frac{\delta H}{\delta w}\right]\right\rangle$$

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We immediately have

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$$\frac{dC(w)}{dt} = -\theta \left\langle \left[\frac{\delta C}{\delta w}, \frac{\delta H}{\delta w}\right], L\left[\frac{\delta C}{\delta w}, \frac{\delta H}{\delta w}\right]\right\rangle = -\theta \left\| \left[\frac{\delta C}{\delta w}, \frac{\delta H}{\delta w}\right] \right\|_L^2$$

- $L = (1 - \alpha^2 \Delta)^s$ to define Sobolev H^s inner product.
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Again, consider Hamiltonian H and Casimir C:

$$\frac{df(w)}{dt} = \underbrace{\left\langle w, \left[\frac{\delta f}{\delta w}, \frac{\delta H}{\delta w}\right]\right\rangle}_{\{f,H\}} - \theta \left\langle \left[\frac{\delta f}{\delta w}, \frac{\delta H}{\delta w}\right], L\left[\frac{\delta C}{\delta w}, \frac{\delta H}{\delta w}\right]\right\rangle$$

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- $L\frac{\delta E}{\delta x}$: conservative, time-reversal symmetric dynamics.
- $M\frac{\delta S}{\delta x}$: dissipative relaxation toward equilibrium.

Section 8

Numerical methods

Let's go back to the magneto-frictional equations, where we have two structures

$$\partial_t A = f, \qquad \frac{d}{dt}\mathcal{E} \leq 0, \qquad \mathcal{H} = \mathcal{H}_0$$

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&= 2\int_{t_n}^{t_{n+1}} (\partial_t A, \boxed{\nabla \times B})\, dt \\
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- He M, Farrell P E, Hu K, Andrews D. B. Helicity-preserving finite element discretization for magnetic relaxation. SISC, 2025.

Introducing two scalars to modify the system:

$$\partial_t A + \lambda_E \boxed{\frac{\delta \mathcal{E}}{\delta A}} + \lambda_H \boxed{\frac{\delta \mathcal{H}}{\delta A}} = f.$$

$$\frac{d}{dt}\mathcal{E} = -F,$$

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$$\begin{pmatrix} \|B\|^2 & (j, B) \\ (j, B) & \|j\|^2 \end{pmatrix} \begin{pmatrix} \lambda_H \\ \lambda_E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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$$\|\boldsymbol{B}\|^2 \|\boldsymbol{j}\|^2 - (\boldsymbol{j}, \boldsymbol{B})^2 > 0,$$

if $\boldsymbol{B} \not\parallel \boldsymbol{j}$ (i.e. $\boldsymbol{j} \times \boldsymbol{B} \neq \boldsymbol{0}$), $\boldsymbol{B} \neq \boldsymbol{0}$ and $\boldsymbol{j} \neq \boldsymbol{0}$.

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- Symmetric positive bracket with Hamiltonian hidden in the kernel (Metriplectic dynamics) $\to$ magneto-frictional system.
- For equilibrium problem: we wish our PDE to be relaxed completely.

$$\boxed{u^*_{\text{variational}} \subset u^*_{\text{dynamics}}}$$

- Numerical methods: we can leverage the hidden structures to develop numerical schemes.

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Thank you!

Mingdong.He@maths.ox.ac.uk