From Classical Thermodynamics to Fluid Motion

Based on Handbook of Mathematical Analysis in Mechanics of Viscous Fluids:
Derivation of Equations for Continuum Mechanics and Thermodynamics of Fluids

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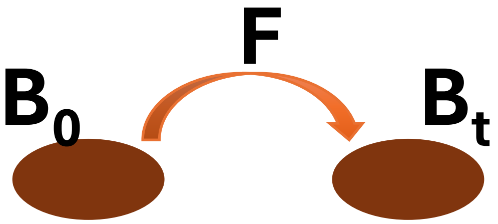
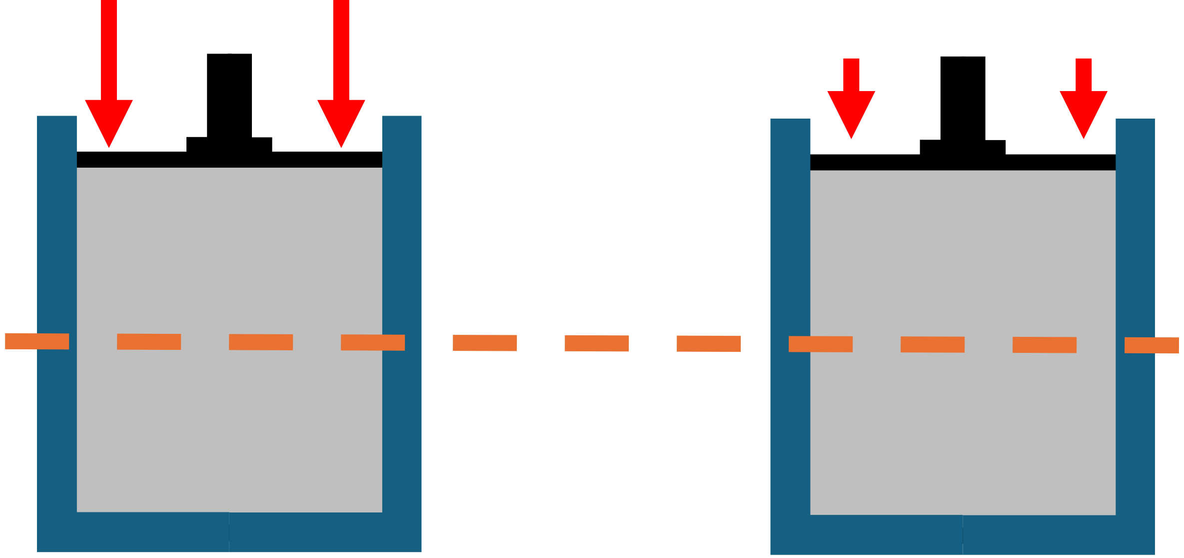
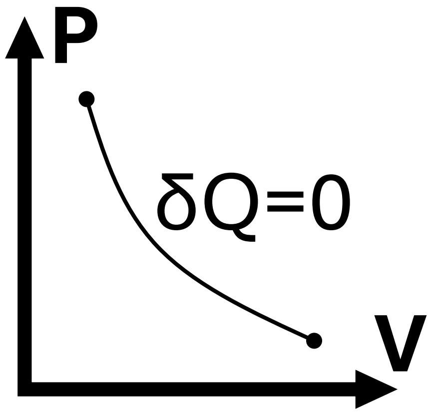
Jakub Cach[1]

[1]Mathematical Institute, Charles University







$$\frac{\mathrm{d}\rho(\boldsymbol{x},t)}{\mathrm{d}t} + \rho(\boldsymbol{x},t)\,\mathrm{div}\,\boldsymbol{v}(\boldsymbol{x},t) = 0,$$

$$\rho(\boldsymbol{x},t)\frac{\mathrm{d}\boldsymbol{v}(\boldsymbol{x},t)}{\mathrm{d}t} = \mathrm{div}\,\mathbb{T}(\boldsymbol{x},t) + \rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t),$$

$$\mathbb{T}(\boldsymbol{x},t) = \mathbb{T}^{\top}(\boldsymbol{x},t),$$

$$\rho(\boldsymbol{x},t)\frac{\mathrm{d}e(\boldsymbol{x},t)}{\mathrm{d}t} = \mathbb{T}(\boldsymbol{x},t) : \mathbb{D}(\boldsymbol{x},t) - \mathrm{div}\,\boldsymbol{j}_q(\boldsymbol{x},t)$$

$$\xi(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)\frac{\mathrm{d}\eta}{\mathrm{d}t}(\boldsymbol{x},t) + \mathrm{div}\left(\frac{\boldsymbol{j}_q(\boldsymbol{x},t)}{\theta(\boldsymbol{x},t)}\right) \geq 0$$

V	=	V
T	≥	T
P	≥	P
δS	≥	δS

$$E = c_V T$$

$$PV = NRT$$

$$\delta S = \frac{\delta Q}{T}$$

$$E = E(S,V)$$

$$T = \frac{\partial E}{\partial S}$$

$$P = -\frac{\partial E}{\partial V}$$

$$e(\boldsymbol{x},t) = e(\eta(\boldsymbol{x},\boldsymbol{t}), \rho(\boldsymbol{x},\boldsymbol{t}))$$

$$e(x,t) = c_v\theta(x,t)$$

$$p_{th}(x,t) = \rho(x,t)R\theta(x,t)$$

$$\mathbb{T} = -p_{\mathrm{th}}(\rho,\theta)\mathbb{I} + \lambda\,(\mathrm{div}\,\boldsymbol{v})\,\mathbb{I} + 2\nu\mathbb{D},$$

$$\boldsymbol{j}_q = -\kappa\nabla\theta,$$

$$e(\boldsymbol{x},t) = e(\eta(\boldsymbol{x},\boldsymbol{t}), \rho(\boldsymbol{x},\boldsymbol{t}))$$

$$\boxed{\begin{aligned} \mathbb{T}_\delta &= 2\nu\mathbb{D}_\delta, \\ m + p_{\text{th}} &= \tilde{\lambda}\,\text{div}\,\boldsymbol{v}, \\ \boldsymbol{j}_q &= -\kappa\nabla\theta, \end{aligned}}$$

$$\mathbb{T} : \mathbb{D} - \text{div}\,\mathbf{j}_e = \rho\dot{e} = \rho\frac{\partial e}{\partial \eta}\dot{\eta} + \rho\frac{\partial e}{\partial \rho}\dot{\rho}$$

$$\dot{\rho} + \rho\,\text{div}\,\mathbf{v} = 0$$

$$\theta := \frac{\partial e}{\partial \eta}, \quad p_{th} := \rho^2\frac{\partial e}{\partial \rho}$$

$$\mathbb{T} : \mathbb{D} - \text{div}\,\mathbf{j}_e = \rho\dot{e} = \rho\theta\dot{\eta} + p_{th}\frac{\dot{\rho}}{\rho} = \rho\theta\dot{\eta} - p_{th}\,\text{div}\,\mathbf{v} \;\Big/ \cdot \frac{1}{\theta}$$

$$\rho\dot{\eta} + \text{div}(\frac{\mathbf{j}_e}{\theta}) = \frac{1}{\theta}\left[\mathbb{T} : \mathbb{D} + p_{th}\,\text{div}\,\mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right]$$

$$\mathbb{T} : \mathbb{D} = (\mathbb{T}^\delta + \frac{1}{3}(\text{tr}\,\mathbb{T})\mathbb{I}) : (\mathbb{D}^\delta + \frac{1}{3}(\text{tr}\,\mathbb{D})\mathbb{I}) = \mathbb{T}^\delta : \mathbb{D}^\delta + m\,\text{div}\,\mathbf{v}$$

$$\rho\dot{\eta} + \text{div}(\mathbf{j}_\eta) = \frac{1}{\theta}\left[\mathbb{T}^\delta : \mathbb{D}^\delta + (m + p_{th})\,\text{div}\,\mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right] = \xi \geq 0$$

Isothermal

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \operatorname{div} \boldsymbol{v},$$

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \operatorname{div} \mathbb{T} + \rho \boldsymbol{b},$$

$$\rho c_V \frac{\mathrm{d}\theta}{\mathrm{d}t} = \mathbb{T} : \mathbb{D} - \operatorname{div} \boldsymbol{j}_q$$

$$\mathbb{T} = -p_{\mathrm{th}}{}^{\star}(\theta, \rho)\mathbb{I} + 2\nu^{\star}(\theta, \rho)\mathbb{D} + \lambda^{\star}(\theta, \rho)(\operatorname{div} \boldsymbol{v})\mathbb{I},$$

$$\boldsymbol{j}_q = -\kappa^{\star}(\theta, \rho)\nabla\theta.$$

$$\mathbb{T} : \mathbb{D} = \underbrace{\cancel{-p_{\mathrm{th}} \operatorname{div} \boldsymbol{v}}}_{\approx 0} + \underbrace{\cancel{\tilde{\lambda} (\operatorname{div} \boldsymbol{v})^2}}_{\approx 0} + \underbrace{\cancel{2\nu \mathbb{D}_\delta : \mathbb{D}_\delta}}_{\approx 0}$$

$$\theta(\boldsymbol{x}, t) = \theta^{\star}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \operatorname{div} \boldsymbol{v},$$

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla \widetilde{p_{\mathrm{th}}}(\rho) + \operatorname{div}\left(2\tilde{\nu}(\rho)\mathbb{D}\right) + \nabla\left(\tilde{\lambda}(\rho) \operatorname{div} \boldsymbol{v}\right)$$

Incompressible?

Isochoric!

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \qquad \operatorname{div} \boldsymbol{v} = 0$$

$$\mathbb{T} : \mathbb{D} = -p_{\mathrm{th}} \operatorname{div} \boldsymbol{v} + \tilde{\lambda} (\operatorname{div} \boldsymbol{v})^2 + 2\nu \mathbb{D}_\delta : \mathbb{D}_\delta$$

$$\mathbb{T} : \mathbb{D} = \mathbb{T}_\delta : \mathbb{D}_\delta$$

$$\frac{1}{\theta}\left[\mathbb{T}^\delta : \mathbb{D}^\delta + (m + p_{th}) \operatorname{div} \mathbf{v} - \mathbf{j}_e \cdot \frac{\nabla\theta}{\theta}\right] = \xi \geq 0$$

$$\mathbb{T} = m\mathbb{I} + \mathbb{T}_\delta$$

$$\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0,$$

$$\operatorname{div} \boldsymbol{v} = 0,$$

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla m + \operatorname{div}\left(2\nu^\star(\theta, \rho)\mathbb{D}\right) + \rho\boldsymbol{b},$$

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\left(e^\star(\theta, \rho) + \frac{1}{2}|\boldsymbol{v}|^2\right) = \operatorname{div}\left(m\boldsymbol{v} + 2\nu^\star(\theta, \rho)\mathbb{D}\boldsymbol{v} + \kappa^\star(\theta, \rho)\nabla\theta\right)$$

$$\boxed{\begin{aligned} &\operatorname{div} \boldsymbol{v} = 0, \\ &\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla m + \operatorname{div}\left(2\nu_\star\mathbb{D}\right) + \rho\boldsymbol{b} \end{aligned}}$$