Mathematical models and numerics for free-boundary MHD and two-phase MHD

Ganghui Zhang

Mathematical Institute, University of Oxford

Reading group, February 13

Overview

Mathematical model

Free-boundary MHD = MHD in one bulk + moving interface.

we consider following incompressible inviscid MHD equations in a moving domain $\mathcal{D}$ and surface tension on the boundary

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \boldsymbol{B} \cdot \nabla \boldsymbol{B} + \nabla P = \boldsymbol{0}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1b}$$

$$\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} = \boldsymbol{0}, \tag{1c}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1d}$$

where $\mathcal{D} = \cup_{t\in(0,T]}\{t\} \times \Omega_+(t)$, $\Omega_+(t) \subset \mathbb{R}^d$, and $P = p + \frac{1}{2}|\boldsymbol{B}|^2$ is the total pressure.

Mathematical model

We require following boundary conditions on the free boundary $\partial\mathcal{D} = \cup_{t\in(0,T]}\{t\} \times \partial\Omega_+(t)$. We will denote $\Gamma(t) = \partial\Omega_+(t)$.

- The domain is material, i.e., $(\partial_t + \boldsymbol{u}\cdot\nabla)|_{\partial\mathcal{D}} \in \mathcal{T}(\partial\mathcal{D})$. Equivalently, the normal velocity of the boundary is determined by the normal velocity of fluid

$$\mathcal{V}_{\boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{n}, \qquad \text{on } \Gamma(t). \tag{2}$$

- Surface tension condition, i.e.,

$$P = \sigma H, \qquad \text{on } \Gamma(t),$$

where σ is a positive coefficient of surface tension and H is the mean curvature of $\Gamma(t)$.

Mathematical model

- The domain is magnetic, i.e. $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on $\Gamma(t)$, where $\boldsymbol{n}$ is the exterior unit normal on $\Gamma(t)$.

We further set the initial condition $\Omega_+(0) \subset \mathbb{R}^3$ is connected domain and function $\boldsymbol{u}^0$ and $\boldsymbol{B}^0$ satisfies

$$\nabla \cdot \boldsymbol{u}^0 = 0, \qquad \nabla \cdot \boldsymbol{B}^0 = 0, \tag{3}$$

and $\boldsymbol{B}^0 \cdot \boldsymbol{n}|_{\Gamma(0)} = 0$.

Physical background

The free-boundary problem originates from the plasma-vacuum free-interface model, which is an important theoretic model both in the laboratory and in astrophysical magnetohydrodynamics.

The basic setting is that the plasma is confined in a vacuum $\Omega = \Omega_+(t) \cup \Omega_-(t)$ with another magnetic field $\widehat{\boldsymbol{B}}$ and

- On $\Omega_+(t)$, the plasma is moving by the free-boundary MHD and therefore its boundary $\Gamma(t) = \partial\Omega_+$ is also moving.
- On $\Omega_-(t)$, the pre-Maxwell system holds in vacuum

$$\nabla \times \widehat{\boldsymbol{B}} = 0, \qquad \nabla \cdot \widehat{\boldsymbol{B}} = 0. \tag{4}$$

- On the interface $\Gamma(t)$, the perfect conducting condition is required that there is no jump in the normal-component:

$$\boldsymbol{B} \cdot \boldsymbol{n} = \widehat{\boldsymbol{B}} \cdot \boldsymbol{n} = 0, \qquad [\![P]\!] = \sigma H. \tag{5}$$

Physical background

- There is a rigid wall wrapping the vacuum region on which the following boundary condition holds on Ω

$$\widehat{\boldsymbol{B}} \cdot \boldsymbol{n}_{\partial\Omega} = \boldsymbol{J}, \tag{6}$$

  where $\boldsymbol{J}$ is the given outer surface current density and $\boldsymbol{n}_{\partial\Omega}$ is the unit outer normal on $\partial\Omega$.

We remark that this setting can be viewed as one-side version of two-phase MHD.

Well-posedness, Ill-posedness and energy-conservation

The well-posedness theory of free-boundary MHD is recently investigated by Gu-Luo-Zhang and Luo-Zhang, follows from the seminal and parallel work of Coutand-Shkoller for Euler free-boundary problem. The main results are listed as follows:

- For zero surface tension case (i.e. $\sigma = 0$). A sufficient condition for the well-posedness is Rayleigh-Taylor's sign condition

$$\nabla P \cdot \boldsymbol{n} < 0, \qquad \text{on } \Gamma(t). \tag{7}$$

  This is proved by Hao and Luo (Arch. Ration. Mech. Anal., 2014).

Well-posedness, Ill-posedness and energy-conservation

- For zero surface tension case (i.e. $\sigma = 0$). If the Rayleigh-Taylor's sign condition is violated, then the problem is ill-posed, showed by Hao and Luo (Commun. Math. Phys. 2020). This is called the Rayleigh-Taylor's instability.
- The positive surface tension can stabilise the problem. The following theorem is proved by Gu, Luo and Zhang (J. Math. Pures Appl. 2024).

Well-posedness, Ill-posedness and energy-conservation

Theorem

Under suitable regularity assumption, and set $\Omega = \mathbb{T}^2 \times (0,1)$*. There exists* $T > 0$ *such that the above free boundary system has a unique strong solution with the energy estimate*

$$\sup_{0 \le t \le T} \mathcal{E}(t) \le C. \qquad (8)$$

Here the model case $\Omega = \mathbb{T}^2 \times (0,1)$ is used for their Lagrangian description in local coordinates. More explicitly, $\partial\Omega = \Gamma_0 \cup \Gamma$, where $\Gamma_0 = \mathbb{T}^2 \times \{0\}$ is a fixed bottom and $\Gamma = \mathbb{T}^2 \times \{1\}$ is the top (moving) boundary we concern.

We remark that these works follow/extend the well-posedness and ill-posedness theory of Euler free boundary problem.

Well-posedness, Ill-posedness and energy-conservation

Theorem (Energy conservation)

Define the total energy

$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega_+(t)} |\boldsymbol{u}|^2 + |\boldsymbol{B}|^2 \, \mathrm{d}\mathcal{L}^d + \sigma \int_{\Gamma(t)} 1 \, \mathrm{d}\mathcal{H}^{d-1}. \tag{9}$$

Then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = 0. \tag{10}$$

Well-posedness, Ill-posedness and energy-conservation

Proof. We compute by Reynolds transport theorem and integration by parts

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega(t)} |\boldsymbol{u}|^2 + |\boldsymbol{B}|^2 \, \mathrm{d}\mathcal{L}^d \\
&= \int_{\Omega_+(t)} (\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{u} + (\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{B}\cdot\boldsymbol{B} \, \mathrm{d}\mathcal{L}^d \\
&= \int_{\Omega_+(t)} \boldsymbol{B}\cdot\nabla\boldsymbol{B}\cdot\boldsymbol{u} - \nabla P\cdot\boldsymbol{u} + \boldsymbol{B}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{B} \, \mathrm{d}\mathcal{L}^d \\
&= -\int_{\Gamma(t)} (\boldsymbol{u}\cdot\boldsymbol{n})\, P \, \mathrm{d}\mathcal{H}^{d-1},
\end{aligned}
$$

Well-posedness, Ill-posedness and energy-conservation

where we use the divergence free property of $\boldsymbol{u}$ and $\boldsymbol{B}$ and its magnetic condition boundary to notice

$$\int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}\mathcal{L}^d = \int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla(\boldsymbol{u} \cdot \boldsymbol{B}) - \boldsymbol{B} \cdot \nabla \boldsymbol{B} \cdot \boldsymbol{u} \, \mathrm{d}\mathcal{L}^d$$
$$= \int_{\Omega_+(t)} -\boldsymbol{B} \cdot \nabla \boldsymbol{B} \cdot \boldsymbol{u} \, \mathrm{d}\mathcal{L}^d.$$

Well-posedness, Ill-posedness and energy-conservation

Then we compute the surface energy is also computed by Reynolds transport theorem as

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma \int_{\Gamma(t)} 1 \,\mathrm{d}\mathcal{H}^{d-1} = \sigma \int_{\Gamma(t)} \mathcal{V}_{\boldsymbol{n}} H \,\mathrm{d}\mathcal{H}^{d-1}.$$

Thus combine with the surface tension equation on the boundary and the property of material domain, we complete the proof.

Weak formulation

We are interested in designing a numerical method that simulate the evolution of free-boundary model of MHD. One of the key questions is:

How to represent the moving interface?

One classical way is the parametric finite element method, that is, we assume there exists a flow map from the reference initial boundary $\widehat{\Gamma}$ to the current boundary

$$\boldsymbol{X} : \widehat{\Gamma} \times (0, T] \to \mathbb{R}^d, \tag{11}$$

and the current boundary is given by $\Gamma(t) = \boldsymbol{X}(\widehat{\Gamma}, t)$ with initial condition $\Gamma(0) = \widehat{\Gamma}$.

Weak formulation

To derive a weak formulation for $(\boldsymbol{u}, \boldsymbol{B}, P)$ and $\boldsymbol{X}$, we introduce the function space

$$\mathbb{U} := [H^1(\Omega_+)]^d,$$

$$\mathbb{P} := L^2(\Omega_+), \quad \widehat{\mathbb{P}} := \left\{ q \in \mathbb{P} : \int_{\Omega_+} q \, \mathrm{d}\mathcal{L}^d = 0 \right\}.$$

We observe that for time-dependent test function supported in $\Omega_+(t)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_+(t)} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d &= \int_{\Omega_+(t)} (\partial_t + \boldsymbol{u} \cdot \nabla)(\boldsymbol{u} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{L}^d \\
&= \int_{\Omega_+(t)} \partial_t \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \partial_t \boldsymbol{v} - (\nabla \cdot \boldsymbol{u})(\boldsymbol{u} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{L}^d \\
&\quad + \int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{H}^{d-1},
\end{aligned}$$

Weak formulation

thus combine with the divergence-free condition of $\boldsymbol{u}$, we can derive

$$
\begin{aligned}
&\int_{\Omega_+(t)} \partial_t \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d \\
&= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_+(t)} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d + \int_{\Omega_+(t)} \partial_t \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \partial_t \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d\right] \\
&\quad - \frac{1}{2}\int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{H}^{d-1} \\
&=: \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \boldsymbol{v})_{\Omega_+} + \frac{1}{2} d_{\Omega_+}(\boldsymbol{u}, \boldsymbol{v}) - \frac{1}{2}\left\langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{u} \cdot \boldsymbol{v} \right\rangle_\Gamma .
\end{aligned}
$$

Weak formulation

Then, by using the general identity for any vector valued function $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} = \frac{1}{2}\left[(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} \cdot \boldsymbol{v}\right] + \frac{1}{2}(\boldsymbol{u} \cdot \nabla)(\boldsymbol{v} \cdot \boldsymbol{w}),$$

we obtain

$$\begin{aligned}\int_{\Omega_+(t)} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d &= \frac{1}{2}\left[\int_{\Omega_+(t)} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{u} \, \mathrm{d}\mathcal{L}^d\right] \\ &\quad + \frac{1}{2}\int_{\Gamma(t)} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{u} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{H}^{d-1} \\ &=: \frac{1}{2} c_{\Omega_+}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \frac{1}{2}\langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{u} \cdot \boldsymbol{v} \rangle_\Gamma\end{aligned}$$

Weak formulation

Thus, the material derivative term can be written as

$$\int_{\Omega_+(t)} (\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v} \; \mathrm{d}\mathcal{L}^d = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}, \boldsymbol{v})_{\Omega_+} + \frac{1}{2} d_{\Omega_+}(\boldsymbol{u}, \boldsymbol{v}) + \frac{1}{2} c_{\Omega_+}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}).$$

Similarly, for the magnetic field term can be written as

$$\int_{\Omega_+(t)} (\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{B} \cdot \boldsymbol{C} \; \mathrm{d}\mathcal{L}^d = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{B}, \boldsymbol{C})_{\Omega_+} + \frac{1}{2} d_{\Omega_+}(\boldsymbol{B}, \boldsymbol{C}) + \frac{1}{2} c_{\Omega_+}(\boldsymbol{B}, \boldsymbol{B}, \boldsymbol{C}).$$

Weak formulation

Subsequently, we will deal with the additional terms. Indeed, by using magnetic boundary condition, we write

$$\int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla \boldsymbol{B} \cdot \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d$$

$$= \int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla(\boldsymbol{B} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{L}^d - \int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{B} \, \mathrm{d}\mathcal{L}^d$$

$$= -\int_{\Omega_+(t)} (\nabla \cdot \boldsymbol{B})(\boldsymbol{B} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{L}^d + \int_{\Gamma(t)} (\boldsymbol{B} \cdot \boldsymbol{n})(\boldsymbol{B} \cdot \boldsymbol{v}) \, \mathrm{d}\mathcal{H}^{d-1}$$

$$- \int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{B} \, \mathrm{d}\mathcal{L}^d$$

$$= -\int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{B} \, \mathrm{d}\mathcal{L}^d$$

$$=: -\ell_{\Omega_+(t)}(\boldsymbol{B}, \boldsymbol{v}, \boldsymbol{B}),$$

Weak formulation

and without any modification, we have

$$\int_{\Omega_+(t)} \boldsymbol{B} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{C} \, \mathrm{d}\mathcal{L}^d = \ell_{\Omega_+(t)}(\boldsymbol{B}, \boldsymbol{u}, \boldsymbol{C}).$$

Finally for the treatment of total pressure term, we pair the term ∇P with the test $\boldsymbol{u}$ and integration by parts and together with the surface tension condition

$$\begin{aligned}\int_{\Omega_+(t)} \nabla P \cdot \boldsymbol{v} \, \mathrm{d}\mathcal{L}^d &= -\int_{\Omega_+(t)} (\nabla \cdot \boldsymbol{v}) P \, \mathrm{d}\mathcal{L}^d + \int_{\Gamma(t)} (\boldsymbol{v} \cdot \boldsymbol{n}) P \, \mathrm{d}\mathcal{H}^{d-1} \\ &=: -(\nabla \cdot \boldsymbol{v}, P)_{\Omega_+} + \sigma \left\langle \boldsymbol{v} \cdot \boldsymbol{n}, H \right\rangle_{\Gamma}.\end{aligned}$$

Weak formulation

The representation or computation of the mean curvature is further depending on how we represent the interface. In the parametric setting, we have a classical formula for mean curvature

$$H\boldsymbol{n} = \Delta_\Gamma \mathrm{Id}, \tag{12}$$

where Δ_Γ is the surface Laplacian and Id is viewed as an identity function on Γ. To consider a weak formulation of surface, we use the surface function spaces

$$\mathbb{V} = H^1(\Gamma), \qquad \mathbb{W} = L^2(\Gamma)$$

Weak formulation

and use the assumption that the normal velocity of interface is controlled by the fluid, we get for any $(\varphi, \boldsymbol{\omega}) \in \mathbb{W} \times \mathbb{V}$

$$\langle \partial_t \boldsymbol{X} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma - \langle \boldsymbol{u} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma = 0,$$
$$\langle H \boldsymbol{n}, \boldsymbol{\omega} \rangle_\Gamma + \langle \nabla_\Gamma \mathrm{Id}, \nabla_\Gamma \boldsymbol{\omega} \rangle_\Gamma = 0.$$

Weak formulation

Sum above derivations up, we finally get the full weak formulation: Find $(\boldsymbol{u}(\cdot,t), \boldsymbol{B}(\cdot,t), P(\cdot,t)) \in \mathbb{U} \times \mathbb{U} \times \widehat{\mathbb{P}}$, and $\boldsymbol{X}(\cdot,t) \in [\mathbb{V}]^3$ and $H(\cdot,t) \in \mathbb{W}$ such that

$$0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u},\boldsymbol{v})_{\Omega_+} + \frac{1}{2}d_{\Omega_+}(\boldsymbol{u},\boldsymbol{v}) + \frac{1}{2}c_{\Omega_+}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) + \ell_{\Omega_+(t)}(\boldsymbol{B},\boldsymbol{v},\boldsymbol{B}) \tag{13a}$$

$$- (\nabla\cdot\boldsymbol{v}, P)_{\Omega_+} + \sigma\left\langle \boldsymbol{u}\cdot\boldsymbol{n}, H\right\rangle_\Gamma, \tag{13b}$$

$$0 = (\nabla\cdot\boldsymbol{u}, q)_{\Omega_+} \tag{13c}$$

$$0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{B},\boldsymbol{C})_{\Omega_+} + \frac{1}{2}d_{\Omega_+}(\boldsymbol{B},\boldsymbol{C}) + \frac{1}{2}c_{\Omega_+}(\boldsymbol{B},\boldsymbol{B},\boldsymbol{C}) - \ell_{\Omega_+(t)}(\boldsymbol{B},\boldsymbol{u},\boldsymbol{C}) \tag{13d}$$

$$0 = \left\langle \partial_t\boldsymbol{X}\cdot\boldsymbol{n}, \varphi\right\rangle_\Gamma - \left\langle \boldsymbol{u}\cdot\boldsymbol{n}, \varphi\right\rangle_\Gamma, \tag{13e}$$

$$0 = \left\langle H\boldsymbol{n}, \boldsymbol{\omega}\right\rangle_\Gamma + \left\langle \nabla_\Gamma \mathrm{Id}, \nabla_\Gamma\boldsymbol{\omega}\right\rangle_\Gamma, \tag{13f}$$

for any $(\boldsymbol{v}, \boldsymbol{C}, q) \in \mathbb{U} \times \mathbb{U} \times \widehat{\mathbb{P}}$ and $(\varphi, \boldsymbol{\omega}) \in \mathbb{W} \times [\mathbb{V}]^3$.

Weak formulation

Theorem

Let $(\boldsymbol{u}(\cdot,t), \boldsymbol{B}(\cdot,t), P(\cdot,t))$, and $(\boldsymbol{X}(\cdot,t), H(\cdot,t))$ is a solution of above system. Then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = 0. \tag{14}$$

Here the energy is defined by

$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega_+} |\boldsymbol{u}|^2 + |\boldsymbol{B}|^2 \,\mathrm{d}\mathcal{L}^d + \sigma \int_{\Gamma(t)} 1 \,\mathrm{d}\mathcal{H}^{d-1}. \tag{15}$$

The proof is essentially the same in the semi-discrete case, so we postpone the proof later.

Finite element approximation

We now consider the finite element discretization of above system. For the moving interface we can use

- The unfitted mesh approach. i.e., the interface and the bulk meshes are completely independent of each other.
- The fitted mesh approach, it ensures that the bulk mesh is aligned with the interface.

Finite element approximation

We start with the semi-discretization. Assume that $\Omega^h \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded Lipschitz domain with polygonal (or polyhedral, respectively) boundary, and $\Gamma^h \subset \Omega^h$ is a $(d-1)$-dimensional polyhedral surface, fully contained in the interior of Ω. We suppose there exist decompositions

$$\overline{\Omega^h} = \bigcup_{j=1}^{J_{\Omega^h}} \overline{o_j}, \qquad \Gamma^h = \bigcup_{j=1}^{J_{\Gamma^h}} \overline{\sigma}_j,$$

where $\{o_j\}_{j=1}^{J_{\Omega^h}}$ and $\{\sigma_j\}_{j=1}^{J_{\Gamma^h}}$ are families of mutually disjoint open d-simplices and $(d-1)$-simplices, respectively.

Finite element approximation

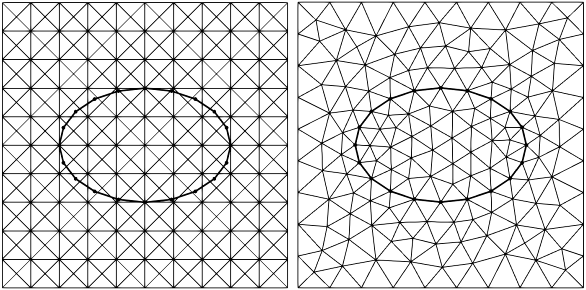

Figure: Unfitted and fitted bulk meshes together with an interface mesh.

Finite element approximation

We define the finite element spaces on Ω^h by

$$S^k(\mathcal{T}^h) := \{\chi \in C(\overline{\Omega}^h) : \chi|_o \in P_k(o), \forall\, o \in \mathcal{T}^h\} \subset W^{1,\infty}(\Omega^h), \;\; k \geq 1,$$
$$S^0(\mathcal{T}^h) := \{\chi \in L^\infty(\Omega^h) : \chi|_o \in P_0(o), \forall\, o \in \mathcal{T}^h\} \subset L^\infty(\Omega^h),$$

where $P^k(o)$ denotes the space of polynomials of degree less than or equal to $k \in \mathbb{N}_{\geq 0}$ on $o \in \mathcal{T}^h$.

Finite element approximation

Given Ω^h and Γ^h, we can separate Ω^h into the interior part exterior part

$$\Omega^h = \overline{\Omega^h_+} \cup \overline{\Omega^h_-}, \tag{16}$$

and defined the restricted finite element space on $\overline{\Omega^h_+}$

$$S^k_+(\mathcal{T}^h) = \left\{ \chi \in S^k(\mathcal{T}^h) : \chi(\boldsymbol{p}_i) = 0 \text{ if } \operatorname{supp}(\chi) \subset \overline{\Omega^h_-},\ i = 1, \dots K_{S^k} \right\}. \tag{17}$$

The inner product on $(\cdot,\cdot)_{\Omega^h_+}$ is naturally induced.

Finite element approximation

We choose the discrete function space tuple $(\mathbb{U}^h(\mathcal{T}^h), \widehat{\mathbb{P}}^h(\mathcal{T}^h)) \subset (\mathbb{U}, \widehat{\mathbb{P}})$ for velocity/pressure such that the usual inf-sup condition is satisfied, meaning there exists a constant $C_0 > 0$ independent of h such that

$$\inf_{\varphi \in \widehat{\mathbb{P}}^h(\mathcal{T}^h)} \sup_{\boldsymbol{\xi} \in \mathbb{U}^h(\mathcal{T}^h)} \frac{(\varphi, \nabla \cdot \boldsymbol{\xi})}{\|\varphi\|_{L^2} \|\boldsymbol{\xi}\|_{H^1}} \geq C_0 > 0. \tag{18}$$

For instance, we can choose the lowest order Taylor-Hood element.

Finite element approximation

The finite element space on Γ^h is defined by

$$\mathbb{V}^h(\Gamma^h)\{\chi \in C(\Gamma^h) : \chi|_{\sigma_j} \in P^1(\sigma_j),\ j = 1, \ldots, J_{\Gamma^h}\} \subset H^1(\Gamma^h),$$

where $P^1(\sigma_j)$ denotes the space of polynomials of degree 1 on σ_j, $j = 1, \ldots, J_{\Gamma^h}$. We define the L^2-inner product and its lumped mass version on polyhedral surface Γ^h as follows

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\Gamma^h} := \int_{\Gamma^h} \boldsymbol{u} \cdot \boldsymbol{v} \ \mathrm{d}\mathcal{H}^{d-1}.$$

Finite element approximation

Based on above construction, the semi-discrete scheme can be described as follows: Given initial approximation $\Gamma^h(0)$ of $\Gamma(0)$ and initial velocity $\boldsymbol{u}^h(0)$, find

$$(\boldsymbol{u}^h(\cdot,t),\boldsymbol{B}^h(\cdot,t),P^h(\cdot,t))\in\mathbb{U}^h(\mathcal{T}^h)\times\mathbb{U}^h(\mathcal{T}^h)\times\widehat{\mathbb{P}}(\mathcal{T}^h),$$

and

$$\boldsymbol{X}^h(\cdot,t)\in[\mathbb{V}^h(\Gamma^h)]^3,\qquad \text{and}\qquad H^h(\cdot,t)\in\mathbb{V}^h(\Gamma^h),$$

such that for any $(\boldsymbol{v}^h,\boldsymbol{C}^h,q^h)\in\mathbb{U}^h(\mathcal{T}^h)\times\mathbb{U}^h(\mathcal{T}^h)\times\widehat{\mathbb{P}}(\mathcal{T}^h)$ and $(\varphi^h,\boldsymbol{\omega}^h)\in\mathbb{V}^h(\Gamma^h)\times[\mathbb{V}^h(\Gamma^h)]^3$.

Finite element approximation

$$0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^h, \boldsymbol{v}^h)_{\Omega_+^h} + \frac{1}{2} d_{\Omega_+^h}(\boldsymbol{u}^h, \boldsymbol{v}^h) \tag{19a}$$

$$+ \frac{1}{2} c_{\Omega_+^h}(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h) + \ell_{\Omega_+^h}(\boldsymbol{B}^h, \boldsymbol{v}^h, \boldsymbol{B}^h) - (\nabla \cdot \boldsymbol{v}^h, P^h)_{\Omega_+^h} + \sigma \left\langle \boldsymbol{u}^h \cdot \boldsymbol{n}^h, H^h \right\rangle_{\Gamma^h}, \tag{19b}$$

$$0 = (\nabla \cdot \boldsymbol{u}^h, q^h)_{\Omega_+^h} \tag{19c}$$

$$0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{B}^h, \boldsymbol{C}^h)_{\Omega_+^h} + \frac{1}{2} d_{\Omega_+^h}(\boldsymbol{B}^h, \boldsymbol{C}^h) + \frac{1}{2} c_{\Omega_+^h}(\boldsymbol{B}^h, \boldsymbol{B}^h, \boldsymbol{C}^h) - \ell_{\Omega_+^h(t)}(\boldsymbol{B}^h, \boldsymbol{u}^h, \boldsymbol{C}^h) \tag{19d}$$

$$0 = \left\langle \partial_t \boldsymbol{X}^h \cdot \boldsymbol{n}^h, \varphi^h \right\rangle_{\Gamma^h} - \left\langle \boldsymbol{u}^h \cdot \boldsymbol{n}^h, \varphi^h \right\rangle_{\Gamma^h}, \tag{19e}$$

$$0 = \left\langle H^h \boldsymbol{n}^h, \boldsymbol{\omega}^h \right\rangle_{\Gamma^h} + \left\langle \nabla_{\Gamma^h} \mathrm{Id}, \nabla_{\Gamma^h} \boldsymbol{\omega}^h \right\rangle_{\Gamma^h}, \tag{19f}$$

Finite element approximation

Theorem

Let $(\boldsymbol{u}^h(\cdot,t), \boldsymbol{B}^h(\cdot,t), P^h(\cdot,t))$, and $(\boldsymbol{X}^h(\cdot,t), H^h(\cdot,t))$ is a solution of above system. Then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^h(t) = 0. \tag{20}$$

Here the energy is defined by

$$\mathcal{E}^h(t) = \frac{1}{2}\int_{\Omega_+^h} |\boldsymbol{u}^h|^2 + |\boldsymbol{B}^h|^2 \, \mathrm{d}\mathcal{L}^d + \sigma \int_{\Gamma^h(t)} 1 \, \mathrm{d}\mathcal{H}^{d-1}. \tag{21}$$

Finite element approximation

Proof. Take test function $\boldsymbol{v}^h = \boldsymbol{u}^h$, $\boldsymbol{C}^h = \boldsymbol{B}^h$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left[\int_{\Omega_+^h} |\boldsymbol{u}^h|^2 + |\boldsymbol{B}^h|^2 \,\mathrm{d}\mathcal{L}^d\right] \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^h, \boldsymbol{u}^h)_{\Omega_+^h} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{B}^h, \boldsymbol{B}^h)_{\Omega_+^h} \\
&= -\frac{1}{2}d_{\Omega_+^h}(\boldsymbol{u}^h, \boldsymbol{u}^h) - \frac{1}{2}c_{\Omega_+^h}(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{u}^h) - \ell_{\Omega_+^h}(\boldsymbol{B}^h, \boldsymbol{u}^h, \boldsymbol{B}^h) + (\nabla\cdot\boldsymbol{u}^h, P^h)_{\Omega_+^h} \\
&\quad - \sigma\left\langle \boldsymbol{u}^h\cdot\boldsymbol{n}^h, H^h\right\rangle_{\Gamma^h} - \frac{1}{2}d_{\Omega_+^h}(\boldsymbol{B}^h, \boldsymbol{B}^h) - \frac{1}{2}c_{\Omega_+^h}(\boldsymbol{B}^h, \boldsymbol{B}^h, \boldsymbol{B}^h) + \ell_{\Omega_+^h(t)}(\boldsymbol{B}^h, \boldsymbol{u}^h, \boldsymbol{B}^h),
\end{aligned}$$

Finite element approximation

then combine with the fact that $d_{\Omega_+^h}(\cdot,\cdot)$ is anti-symmetric, $c_{\Omega_+^h}(\cdot,\cdot,\cdot)$ is anti-symmetric with last two components, and take the test function $q^h = P^h$ in the second equation, we arrived at

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left[\int_{\Omega_+^h} |\boldsymbol{u}^h|^2 + |\boldsymbol{B}^h|^2 \,\mathrm{d}\mathcal{L}^d\right]$$

$$= -\sigma \left\langle \boldsymbol{u}^h \cdot \boldsymbol{n}^h, H^h \right\rangle_{\Gamma^h}.$$

Finite element approximation

Subsequently, we will use two interface equations by computing the discrete surface energy by Reynolds transport theorem and taking test function $\boldsymbol{\omega}^h = \partial_t \boldsymbol{X}^h$ and $\varphi^h = H^h$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma^h(t)} 1 \ \mathrm{d}\mathcal{H}^{d-1} &= \int_{\Gamma^h(t)} \nabla_{\Gamma^h}\mathrm{Id} \cdot \nabla_{\Gamma^h}\partial_t \boldsymbol{X}^h \ \mathrm{d}\mathcal{H}^{d-1} \\
&= \left\langle \nabla_{\Gamma^h}\mathrm{Id}, \nabla_{\Gamma^h}\partial_t \boldsymbol{X}^h \right\rangle_{\Gamma^h} \\
&= -\left\langle H^h \boldsymbol{n}^h, \partial_t \boldsymbol{X}^h \right\rangle_{\Gamma^h} \\
&= \left\langle \boldsymbol{u}^h \cdot \boldsymbol{n}^h, H^h \right\rangle_{\Gamma^h}.
\end{aligned}
$$

To summarise, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^h = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left[\int_{\Omega_+^h} |\boldsymbol{u}^h|^2 + |\boldsymbol{B}^h|^2 \ \mathrm{d}\mathcal{L}^d\right] + \sigma \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma^h(t)} 1 \ \mathrm{d}\mathcal{H}^{d-1} = 0.
$$

Summary

Questions:

- What about full discrete case?
- How about use finite element exterior calculus such as Hdiv and Hcurl space?
- How about the two-phase MHD model?

Thanks for Listening! Any Questions?